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A RELAY MEMORY FOR A THYRATRON COUNTER

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ABSTRACT. The paper describes an automatic mechanism consisting of sixteen interconnected relays and capable of carrying out a complicated cycle of operations in correct sequence in less than half a second. The apparatus is used in conjunction with a valve amplifier and an automatic thyratron counter for the analysis of α-particle groups by means of a magnetic focusing method. The relay mechanism arranges for alternate comparative counts of α particles to be made under two different sets of experimental conditions, *A* and *B*. At the end of a 9-sec. counting period under condition *A*, a relay memory *A* records and remembers the thyratron-counter readings; a change-over of experimental conditions from *A* to *B* takes place, and another relay memory *B* restores the thyratron counter to the state at which it stopped at the end of the previous 9-sec. *B* count, from which point it continues to count under experimental condition *B*. In the meantime, memory *A* continues to remember its reading for 10 sec. until the next change-over, when conditions are restored to state *A* again, the cycle being repeated throughout the course of the experiment.

§ 1. GENERAL DESCRIPTION

IN a paper* published elsewhere, Lewis and Bowden describe an experiment which was carried out to investigate the fine structure of the α-particle groups from thorium *C* and thorium *C'*. As in previous experiments,† the α-particle groups were analysed by means of an annular magnetic field, and individual particles were counted by means of a valve amplifier and a thyratron scale-of-two automatic counter; for certain reasons, however, an entirely different method of working was adopted. Previously, the analysis had been carried out by focusing the various groups in turn on to the ionization-chamber slit, and, by suitably adjusting the strength of the magnetic field, exploring the region under investigation; the velocities of the particles in the various groups were then determined from the

* *Proc. R.S.* A **145**, 854 (1934).

† Rutherford, Wynn-Williams, Lewis and Bowden, *Proc. R.S.* A **139**, 617 (1933), and Rutherford, Lewis and Bowden, *Proc. R.S.* A **142**, 347 (1933).

measurements of the strengths of the magnetic fields. In the more recent experiment, however, a method was adopted of measuring directly, in electron-volts, the energy-difference between two particle groups. This was done by maintaining the magnetic field as constant as possible at a value which would normally focus particles whose energies were equal to the mean value for the two groups, and applying alternately an accelerating and a retarding potential of + 10,000 V. and − 10,000 V. to the source. In this way, particles belonging to the two groups were in turn brought to a focus on the ionization chamber-slit, and alternate counts were made of the numbers of particles in the two groups. As Lewis and Bowden point out in their paper, this method has the great advantage of enabling one to pass instantly from one group to another, and also of enabling the energy-difference between the two groups to be measured directly in terms of the electric field, without involving a measurement of the strength of the magnetic field, provided the magnetic field is kept sufficiently steady during this process. In order that this latter condition may be realized, it is necessary to take alternate readings for the positive and negative potentials (which will be referred to in this paper as *experimental conditions A and B*), as frequently as possible, so that errors due to any slight inconstancy of the magnetic field will be negligible.

It is, of course, possible to carry out the change-over from condition *A* to condition *B* by hand, to take a large number of alternations in this manner, and to add up the two series of observations. Such a process, however, proved to be extremely tedious. Moreover, to enable the useful counting-time to be large compared with that required for the change-over operations, which would occupy several seconds, alternate counts of about half a minute would be necessary, and this would necessitate the change-overs being less frequent than is desirable. For this reason it is highly desirable in an experiment of this kind to devise and employ some form of apparatus to carry out all the operations automatically, so that this may be done with greater precision and speed and without the fatigue which is occasioned during the repetition of cycles of operations every half minute over very long periods.

Some idea of the operations involved in taking only a single pair of such observations, i.e. two half-minute counts, may be gathered from the following programme of instructions covering one cycle of events occupying a little over a minute. (1) Start counter and count for 30 sec. under condition *A*. (2) Switch off counter. (3) Record counter readings. (4) Reset counter. (5) Reverse electric field. (6) Start counter and count for 30 sec. under condition *B*. (7) Switch off counter. (8) Record counter readings. (9) Reset counter. (10) Reverse electric field.

The object of the present paper is to describe a form of automatic apparatus which has been devised for the purpose referred to and has proved highly efficient. The apparatus consists of sixteen relays, of the pattern used in telephone exchanges, interconnected in groups in such a way that it is impossible for them to operate in any but the correct sequence, and that a given relay operation cannot commence until the preceding operation has been correctly and completely performed. As will be seen later, there are nine stages in a change-over from one condition to the

other, and all the operations involved in the complete change-over are carried out by the relays in well under half a second. By this means it is possible to alternate the counts every 10 sec., 9 sec. being allowed for the count and 1 sec. for the complete change-over, so that there is a margin of at least half a second. It is estimated by Lewis and Bowden that the error involved on account of drift of the magnetic field during such short counts is not more than one part in two million.

As is well known, automatic relay mechanisms are extensively employed in connexion with automatic telephone exchanges and in many other branches of electrical engineering. In the past, however, little use seems to have been made of apparatus of this kind in connexion with purely physical problems. It is hoped that the present paper will serve to illustrate how a fairly complicated sequence of events can be carried out by such apparatus. There are many other physical problems to which relay mechanisms could be applied with great advantage, and the devising of suitable circuits is not very difficult when once the underlying principles have been grasped.

In what follows, it is assumed that the reader is acquainted with the principle of operation of the thyratron scale-of-two counter, which has been previously described*. It may be recalled, however, that with such a counter the natural numbers are recorded on a scale of two. Multiples of 8 or 2^3 are recorded on the dial of a mechanical counting meter, while numbers between 0 and 7, which are given by the addition of combinations of 2^0, 2^1 and 2^2, or 1, 2 and 4, are read off from the positions of the three arcs in the six thyratrons *AB*, *CD* and *EF*. Arcs in *B*, *D* and *F* respectively denote 1, 2 and 4, and in *A*, *C* and *E*, 0. By the use of two mechanical meters the multiples of 8 relating to the two sets of experimental conditions can be permanently recorded; but numbers between 0 and 7, i.e. the uncompleted eights, since they are represented only as arc-combinations, would be lost during a change-over, unless they could be recorded and remembered by some device. This is effected by means of two relay memories.

The general scheme consists in arranging three groups of relays as follows: (1) a group, called the "control" group, whose functions are (i) to carry out in correct sequence the nine operations involved in the change-over from one experimental condition to the other on receipt of an impulse from a clock every 10th sec., and (ii) to restart the counting process exactly 1 sec. later; (2) two other groups, called "memory *A*" and "memory *B*," associated with the thyratron counter and interconnected with the control group, which come into operation alternately. When operated by the control group, these latter serve to record the positions of the thyratron arcs, to remember the reading for 10 sec., and subsequently to reset the thyratron counter to this position so as to continue with the count. In this way, at a change-over, the combination of arcs at the end of an *A* counting-period is remembered by the *A* memory. In the meantime the thyratron counter is being used for a *B* count. At the next change-over, the *B* memory records and remembers the positions of the arcs at the end of the *B* counting period, while the *A* memory restores the arcs to the positions they occupied at the end of the preceding *A* count

* C. E. Wynn-Williams, *Proc. R.S.* A **136**. 312 (1932).

and so enables them to continue to count from this point under experimental condition *A*, and *vice versa*.

The arrangement used is shown in the two figures. Figure 1 illustrates the general principle, while figure 2 shows the relay circuits. In figure 1, for simplicity, only the anodes and the grids of the thyratrons are shown, the rest of the counter circuit being exactly as illustrated in figure 5 of the original paper on the scale-of-two counter*.

§ 2. MODE OF OPERATION

In each of the memory groups, three relays, 1*a*, 2*a* and 4*a*, and 1*b*, 2*b* and 4*b*, figure 1, are associated with the three thyratrons *B*, *D* and *F*, which, when in the arcing condition, indicate respectively that 1, 2 and 4 are to be added to the meter reading. When the in relay *Ia* or *Ib* of a memory group is operated, these three relays are momentarily connected to the anodes of their associated thyratrons. If there are arcs in any of these thyratrons the corresponding relays are operated and locked, so as to record and remember the state of the thyratrons, i.e. the

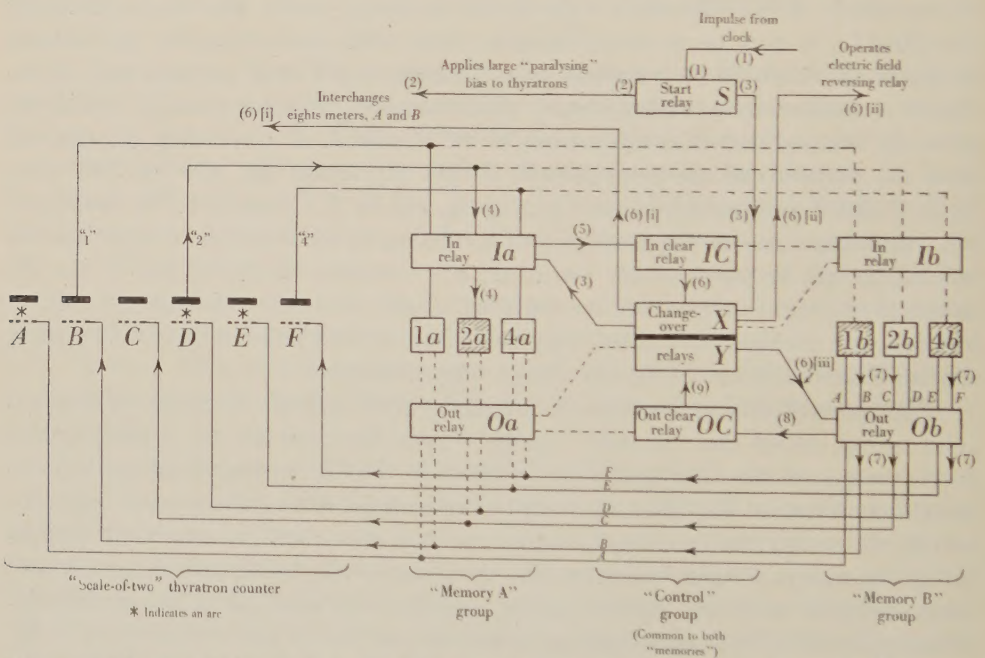

Figure 1. Memory *A* shown recording 2 from thyratron *D*; as soon as this is completed, control group causes memory *B* (which had been remembering 5 as 1 + 0 + 4) to transfer the arcs to thyratrons *B*, *C* and *F*, corresponding to 5. The numbers opposite the arrows correspond to those of the accompanying programme of events.

positions of the arcs. When the out relay *Oa* or *Ob* is operated, on the other hand, a large positive impulse is sent along three of the six wires attached to the six thyratron grids, so as to bring the arcs from whatever position they occupy to a new

* *Loc. cit.*

starting position, which is identical with some previous stopping position which had been recorded and remembered by the 1, 2 and 4 relays of either memory. The decision as to which grid of each of the three pairs of thyratrons *AB*, *CD*, *EF* receives the impulse, is governed by the state of the associated relay; for example, if the 2 relay is locked, *D* receives the impulse, and if it is unlocked, *C*.

The control group consists of six relays, a start relay *S*, a pair of change-over relays *X* and *Y*, an in clear relay *IC*, an out clear relay *OC*, and a meter-guard relay *G* (not shown in figure 1). The function of these relays is to bring about the required alternations of the experimental conditions every 10 sec. Conditions are reversed by relays *X* and *Y*. Condition *A* is maintained by *X* and *Y* remaining unlocked during the 9-sec. counting-period, while condition *B* is maintained by *X* and *Y* remaining locked during the period.

The following programme of events indicated by the lines and numbered arrows of figure 1, taken in the order set out, will make the mode of operation clear. Assume that a count is in progress under experimental condition *A*, with a positive potential on the source.

At the tenth second. (1) A clock impulse operates the start relay *S*. (2) The thyratrons are rendered inoperative by the application of a large negative grid bias. Suppose that the arcs are in thyratrons *A*, *D* and *E*, corresponding to 2, when counting ceases. (3) An impulse is sent via relay *X* of the control group, to the in relay, *Ia* of memory *A*. (4) The in relay *Ia* momentarily connects relays 1*a*, 2*a* and 4*a* of memory *A* to the thyratron anodes. Relay 2*a* locks, but not 1*a* or 4*a* as there are no arcs in thyratrons *B* and *F*. (5) The in relay *Ia* signals the end of the recording operation by operating the in clear relay *IC* of the control group. (6) The in clear relay causes one of the change-over relays *X* to lock. This causes [i] eights meter *B* to be substituted for meter *A*, [ii] a relay (not shown) to reverse connexions to the 10,000 V. potential-supply; [iii] an impulse to be sent to the out relay *Ob* of the *other* memory group *B*. (7) The out relay *Ob* causes positive impulses to be sent along three grid wires, the selection of which depends on which of the 1*b*, 2*b* and 4*b* relays had been locked during the previous change-over. Suppose these were 1*b* and 4*b*; then the arcs are transferred from positions *A*, *D* and *E* to positions *B*, *C* and *F*. (Later on, at the eleventh second, relays 1*b* and 4*b* are unlocked, so as to forget, as they have now served their purpose for the time being.) (8) The out relay *Ob* signals the end of the restoring process by operating the out clear relay *OC* of the control group. (9) The out clear relay *OC* causes the other change-over relay *Y* to lock, preventing further relay movements.

All the above operations (1) to (9) are carried out in rapid succession in less than half a second.

At the eleventh second. The start relay *S* is unlocked by the clock; the large negative bias applied to the thyratrons to paralyse them during the change-over is removed, and the count proceeds under experimental condition *B*, corresponding to a negative potential on the source, multiples of eight being now recorded on the eights meter *B*. This condition is maintained throughout the next 9 sec. by relays *X* and *Y* remaining locked during this period.

At the twentieth second. A change-over back to normal (condition *A*) takes place, the sequence of events being much as before but with the operations of the memory groups *A* and *B* interchanged. Operations (1) and (2) are as before. In (3), (4) and (5), on account of relays *X* and *Y* now being locked instead of free as before, an impulse is directed to the in relay *Ib* instead of *Ia*, and memory *B* therefore carries out the recording instead of memory *A*, as is indicated by the broken lines in figure 1. At (6) relay *X* is *unlocked*, reversing the experimental conditions from state *B* back to state *A*; and at (6) [iii], (7) and (8), on account of relay *Y* being locked, instead of unlocked as before, the out relay *Oa* of memory *A* is operated, causing memory *A* to carry out the restoring operation previously performed by memory *B*. At (9) relay *Y* is *unlocked*, and the count under condition *A* is resumed at the twenty-first second.

A cycle, of two 9-sec. counts under the two experimental conditions and two complete change-overs, has thus been completed in 20 sec. This process continues, and the total numbers of particles counted under the two sets of conditions are given finally on the dials of the two eights meters*.

It might appear that this method of achieving the desired result is unnecessarily complicated†. It should, however, be emphasized that a system of interconnected relays such as this is capable of carrying out the complicated sequence of operations with precision in a far shorter time than could be done by any other electromechanical device and, in any case, in a fraction of the time required by a human observer. Moreover, the speed and precision of the operations do not fall off with time during a prolonged run.

§ 3. OPERATION OF RELAY CIRCUITS

The action of the relays in the previously described programme of events following the tenth second and numbered (1) to (9) will now be described in detail. The sectional numbers given below correspond to those of the previous programme, and the various circuit elements are designated by the letters (*a*) to (*h*) so that they may be easily traced in figure 2.

(1) *Circuit* (*a*). The clock closes and opens contact *P* every second, and *Q* every tenth second. At the tenth second, circuit of start relay *S* is completed via +, winding *S*, *Q* and −, and relay *S* is energized. All *S* contacts are operated. When *P* and *Q* subsequently open together, about half a second later, *S* is still kept in the operated state since it is locked via +, the other winding *S'*, resistance

* Of the two final uncompleted eights, one can be read directly from the final positions of the thyratron arcs, while the other is registered on one of the two memories and can be read from the relays themselves, or from the thyratron arcs by arranging a change-over.

† An experiment of this kind could, of course, be carried out by means of two separate thyratron counters, so that the relay memories *A* and *B* could be dispensed with. Even so, however, several relays would be required to effect the rapid change-over, and the inclusion of the memory groups entails far less complication than a second complete counter would. Further, the expense involved in the construction of the relay apparatus is far less than that of six additional thyratrons and their associated apparatus, and its useful life is much greater. Finally, it is always preferable to take sets of experimental observations with the same piece of apparatus when they have to be compared with one another.

$R.1$, $S.2$ and $-$. It remains in this state throughout the remaining stages. (2) *Circuit* (*e*). S.1 closes before any other S contact, and applies a large negative potential across $R.2$ to increase the thyratron bias sufficiently to paralyse the counter, thus stopping the count and guarding the counter from surges. (3) *Circuit* (*a*). The closing of $S.2$ completes the circuit of the in relay *Ia*, via $+$, $OC.1$, $Y.3$,

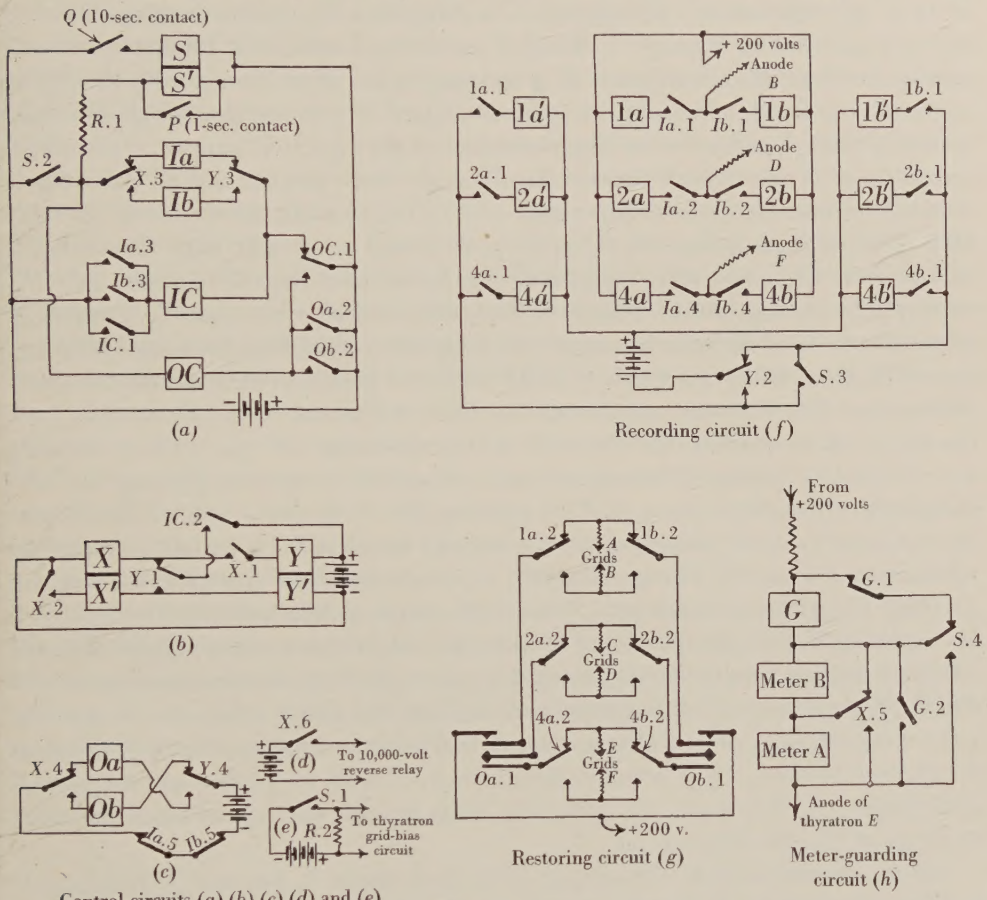

Figure 2*.

winding *Ia*, $X.3$, $S.2$ and $-$. Relay *Ia* is energized†. (4) *Circuit* (*f*). Contacts $Ia.1$, $Ia.2$ and $Ia.4$ therefore close, and rather later, contact $Ia.3$ closes in circuit (*a*). The former complete the circuits of relays 1*a*, 2*a* and 4*a*. Suppose there is an arc in thyratron D; relay 2*a* is energized via + 200 V., winding 2*a*, $Ia.2$, resistance,

* For simplicity, the usual convention of separating relays and their contacts has been adopted. It will be understood that when any relay, such as S, is actuated, all contacts designated by the relay letter, such as $S.1$, $S.2$, $S.3$ and $S.4$, operate, although they may appear in the figure in different circuits. When a relay has two windings the same letter is used for both; thus, windings S and S' both belong to relay S. A common battery suffices for all the circuits, but it is less confusing to show a separate battery for each.

† Note that had relays X and Y been *locked* prior to the change-over period, $X.3$ and $Y.3$ would have been down and *Ib* would have operated instead of *Ia*.

anode of thyratron D, arc and cathode. Relay $2a$ is actuated, contact $2a.1$ operates, and also contact $2a.2$ in circuit (g). Relay $2a$ is now locked independently of $Ia.2$, via the circuit +, the other winding $2a'$, $2a.1$, $S.3$, $Y.2$ and −. (5) *Circuit* (a). Late closing of $Ia.3$ completes the circuit of the in clear relay via +, $OC.1$, winding IC, $Ia.3$, and −. Relay IC actuated, and contact $IC.1$ now locks IC independently of $Ia.3$. (6) *Circuit* (b). Closing of $IC.2$ completes the circuit of relay X via +, $IC.2$, $Y.1$, winding X, and −. Relay X is actuated and all X contacts operate*, causing *in circuit* (h) operation of $X.5$ to transfer the short-circuit from the eights meter B to meter A; *in circuit* (d) closure of contact $X.6$ to energize a high-potential reversing relay, which reverses the connexions of the 10,000 V. supply; *in circuit* (a) operation of $X.3$ to break the circuit of Ia, so that relay Ia and all Ia contacts return to normal; *in circuit* (c) operation of $X.4$ to complete the circuit of out relay Ob† via +, $Y.4$, winding Ob, $X.4$, $Ia.5$, $Ib.5$ and −, so that relay Ob actuated, and contacts $Ob.1$ and $Ob.2$ close, the latter rather later than the former; (7) *Circuit* (g). The closing of $Ob.1$ connects $1b.2$, $2b.2$ and $4b.2$ to + 200 V. Supposing relays $1b$ and $4b$ to be locked, contacts $1b.2$ and $4b.2$ would be down, and $2b.2$ up; hence the grids of thyratrons B, C and F receive a positive potential, via the series resistances. The arcs are transferred to these thyratrons. (8) *Circuit* (a). Late closing of $Ob.2$ completes the circuit of the out clear relay OC via +, $Ob.2$, winding OC, $S.2$, and −. Relay OC is actuated and contact $OC.1$ operates, locking relay OC independently of $Ob.2$, via +, $OC.1$, winding OC, $S.2$, and −; $OC.1$ also breaks the circuit of relay IC, which now releases, and contacts $IC.1$ and $IC.2$ return to normal. (9) *Circuit* (b). The opening of $IC.2$ breaks the short-circuit across winding Y; relay Y operates in series with X, and both relays remain locked‡. In circuit (c), operation of $Y.4$ breaks the circuit of out relay Ob, which releases; all Ob contacts return to normal, and positive potential is removed from the thyratron grids. In circuit (f), operation of $Y.2$ maintains a locking circuit for relays $1a$, $2a$ and $4a$, and arranges for the circuit of relays $1b$, $2b$ and $4b$ to be broken when $S.3$ opens at the eleventh second; i.e. it arranges that memory A shall remember and memory B shall forget. At this stage, the apparatus waits for the start signal from the clock to arrive at the eleventh second.

At the eleventh second. Circuit (a). The clock closes P, but not Q. Winding S' is short-circuited by P, but the battery is protected by resistance $R.1$. Relay S now unlocks, and all S contacts return to normal. $S.2$ releases the out clear relay OC in circuit (a). In circuit (f), $S.3$ releases all the B memory relays, but the A memory relays are still locked via $Y.2$, which is now down. Finally§, in circuit (e), $S.1$ removes the high grid bias, and the thyratrons begin counting.

During the next 9 sec. relays X and Y remain locked in series, all their contacts

* Note that relay Y is now in series with X via $X.1$, but cannot operate until subsequent opening of $IC.2$, at stage (9), breaks the short-circuit $X.1$, $Y.1$, $IC.2$, winding Y, and $X.1$.

† Note that (i) relay Ob cannot operate until $Ia.5$ is right back to normal, indicating that the recording process is completed; and (ii) if X and Y were *locked* prior to the change-over, $X.4$ and $Y.4$ would have been down and Oa would have operated instead of Ob on the operation of $X.4$. Compare with the footnote to stage (3).

‡ See footnote to stage (6).

§ $S.1$ is so adjusted as to close before and to open after the operation of $S.2$, $S.3$ and $S.4$.

being in the operated state. At the twentieth second, *Ib* operates instead of *Ia*, and *Oa* instead of *Ob*, as has been mentioned in the footnotes to stages (3) and (6). At (6), closing of $IC.2$ connects the other winding X' of relay X in parallel with winding X, via +, $IC.2$, $Y.1$, winding X', $X.2$, and −; the magnetic field of windings X and X' neutralize one another and relay X releases, but relay Y is still locked via +, $IC.2$, $Y.1$, winding Y', and −. At (9), the opening of $IC.2$ breaks this circuit and relay Y unlocks, all X and Y contacts now having reverted to normal for the next count. Corresponding contacts in the two memory groups carry out similar operations during the alternate counts, and a similar cycle can be followed for events (1) to (9) with the letters *a* and *b* interchanged in the foregoing. It should be borne in mind that relays X and Y start this time by being *locked*.

Meter guard relay, circuit (*h*). The two eights meters are connected in series in the anode circuit of thyratron E, and one or other is normally rendered inoperative by being short-circuited by contact $X.5$. During a change-over the anode current of thyratron E may suffer interruption, which would cause spurious eights to be recorded; this is prevented by arranging that both meters shall be short-circuited, by the operation of $S.4$, during the whole of the change-over process. If, however, there is an arc in thyratron E when $S.4$ subsequently returns to normal, one meter will immediately score a spurious eight. To prevent this, a guard relay G, which is normally short-circuited by $G.1$ and $S.4$ in series, is employed. When $S.4$ is operated the short-circuit is broken, and if there is an arc in thyratron E relay G is energized and places an additional short-circuit across the two meters via $G.2$; it also breaks its own short-circuit at $G.1$. After the return of $S.4$ to normal at the commencement of a counting-period both meters are still short-circuited by $G.2$; at some stage, the arc moves from thyratron E to thyratron F causing relay G to release and re-short-circuit itself at $G.1$, and to render one meter operative by the opening of $G.2$. This state continues during the rest of the counting-period. Thus relay G diverts an initial spurious eight impulse away from the meters.

538.122

MAGNETIZED ELLIPSOIDS AND SHELLS IN A PERMEABLE MEDIUM

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ABSTRACT. The relation between the magnetic fields round a uniformly magnetized ellipsoid when in a vacuum and when immersed in a permeable medium is investigated. The probability that the field round a thin normally magnetized shell is independent of the permeability of the medium surrounding it is discussed.

It has been shown in a former communication* that if the magnetic force at an external point due to a uniformly magnetized sphere surrounded by a medium of
μ constant relative permeability μ is compared with the magnetic force due to a sphere of the same radius and magnetization surrounded by a vacuum, the forces at corresponding points throughout the field will correspond in direction and be proportional in magnitude. For spheres the ratio of the forces was shown to be $3/(2\mu + 1)$.

More recently, this result has been extended by Prof. Leigh Page† to the case of a spheroid uniformly magnetized parallel to its axis of symmetry, and the ratio of the forces has been determined as a function of the eccentricity of the spheroid. It has become of interest, therefore, to inquire whether the same result is capable of still further extension, and if so with what limitations, if any.

It is easy to show that the result holds good for the case of a body with uniform
I magnetization I and of such a shape that the demagnetizing force within it due to
λ its magnetization is uniform and collinear with I and so may be denoted by $-\lambda I$.

If such a body is surrounded by a vacuum the magnetic force at either an external or an internal point is well known to be that due to a surface distribution $+ I \cos\theta$,
θ where θ is the angle between the positive directions of I and of an outward-drawn normal.

If the body is surrounded by a medium of constant relative permeability μ we shall proceed to show that the magnetization produced in this medium will give
α rise to the same magnetic force as a surface distribution of density $-\alpha I \cos\theta$, where α has a suitable constant value. For on this assumption the magnetic force in the positive direction of I at an internal point will be $-\lambda I(1-\alpha)$ and at a point just inside the surface of the body this can be resolved into $-\lambda I(1-\alpha)\sin\theta$ tangentially and $-\lambda I(1-\alpha)\cos\theta$ normally outwards. At an adjacent point in

* *Proc. Phys. Soc.* **45**, 82 (1933).
† *Phys. Rev.* **44**, 112 (1933).

the medium, while the tangential force is unaltered, the two surface distributions $+ I \cos\theta$ and $- \alpha I \cos\theta$ increase the normal outward force by $4\pi I \cos\theta$ and $- 4\pi\alpha I \cos\theta$ respectively, and it thus becomes $I(1-\alpha)\cos\theta(4\pi - \lambda)$. The boundary conditions which must be satisfied are that at the surface of separation the tangential components of the internal and external magnetic force and the normal components of the internal and external magnetic induction shall be respectively equal. Of these, the former is satisfied identically. As to the latter, since the magnetic induction within the body is the sum of $4\pi I$ and the magnetic force $- \lambda I(1-\alpha)$, its normal component is $I\cos\theta[4\pi - \lambda(1-\alpha)]$, and since the normal component at a point in the medium is the product of μ and the normal magnetic force, it follows that

$$I\cos\theta[4\pi - \lambda(1-\alpha)] = \mu I\cos\theta(1-\alpha)(4\pi - \lambda).$$

This is satisfied for all values of θ if $4\pi - \lambda(1-\alpha) = \mu(1-\alpha)(4\pi - \lambda)$ and hence our assumption is justified and the value of α is given by

$$1 - \alpha = 4\pi/(4\pi\mu - \mu\lambda + \lambda), \text{ or } \alpha = (\mu - 1)(4\pi - \lambda)/(4\pi\mu - \mu\lambda + \lambda).$$

The second boundary condition may also be obtained, as in the case of a sphere*, by equating the surface density $- \alpha I\cos\theta$ due to the magnetization of the medium and the product of the susceptibility κ and the normal inward magnetic force in the medium. Thus κ

$$- \alpha I\cos\theta = - \kappa I\cos\theta(4\pi - \lambda)(1-\alpha), \text{ or } \alpha/(1-\alpha) = \kappa(4\pi - \lambda).$$

Remembering that $\kappa = (\mu - 1)/4\pi$, this gives the same value for α as before.

We may note, again as in the case of a sphere, that the induction within the body is given by $4\pi I - \lambda I$ if the body is in a vacuum and by $4\pi I - \lambda(1-\alpha) I$ if the body is surrounded by the medium. It is clear that the ratio of the latter to the former must be the product of μ and the constant ratio of the magnetic forces at corresponding points in the latter and the former case. Thus

$$\mu(1-\alpha) = (4\pi I - \lambda I + \alpha\lambda I)/(4\pi I - \lambda I),$$

which again leads to the same value of α.

Following the notation employed by Page, let γ be the ratio of the field-strength for a vacuum to that for the medium, and let r be the quotient $(\gamma - 1)/(\mu - 1)$. γ r Then

$$\gamma = 1/(1-\alpha) = \mu - (\mu - 1)\lambda/4\pi$$

and

$$r = 1 - \lambda/4\pi.$$

Now it has been pointed out by Maxwell† that the most general form for a body in which uniform magnetization produces uniform demagnetizing force is an ellipsoid, and further that if the demagnetizing force is collinear with the magnetization their direction must be that of one of the axes.

Let a, b, c be the axes of the ellipsoid. Then if the magnetization is along the axis a the corresponding value of λ, which we may denote by λ_1, is† a, b, c λ_1

$$2\pi abc\int_0^\infty \frac{du}{(a^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

* L. R. Wilberforce, *loc. cit.*

† *Electricity and Magnetism*, 2, 67–9 (3rd edition).

If the ellipsoid is now surrounded by a medium of permeability μ, the lines of force in the medium will be identical with those in the vacuum which it replaces, but the magnetic force at each point will be divided by $\mu - (\mu - 1)\lambda_1 4\pi$. This is equivalent to the statement that the magnetic field in the medium due to an ellipsoid with magnetization I along the axis a is the same as that in a vacuum due to the same ellipsoid with magnetization $I/[\mu - (\mu - 1)\lambda_1 4\pi]$ along the same axis.

Similar results follow if the ellipsoid is magnetized along the axis b or the axis c, λ_2, λ_3 the quantity λ_1 being replaced by λ_2 in the one case and λ_3 in the other, where

$$\lambda_2 = \int_0^\infty \frac{du}{(b^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

and

$$\lambda_3 = \int_0^\infty \frac{du}{(c^2+u)\sqrt{(a^2+u)(b^2+u)(c^2+u)}}.$$

It follows that if the ellipsoid is uniformly magnetized with intensity I but l, m, n non-axially, so that the direction cosines of I referred to the axes are l, m, n and if it is surrounded by the medium, the resultant field will be the same as that in a vacuum due to an ellipsoid of the same dimensions but so magnetized that its axial components are

$$Il/[\mu - (\mu - 1)\lambda_1/4\pi], \quad Im/[\mu - (\mu - 1)\lambda_2/4\pi], \quad In/[\mu - (\mu - 1)\lambda_3/4\pi].$$

This resultant magnetization is of course uniform, but it differs in direction from that of the original ellipsoid; thus the two fields no longer correspond in direction and therefore cease to be everywhere proportional in magnitude. Hence the solution for this comparatively simple case does not bear out Page's generalization that "the effect of immersing a permanently magnetized body of any shape in a magnetic medium of constant permeability is to reduce the field everywhere in the same ratio, this ratio being a function of the permeability of the medium and of the geometrical configuration of the body alone."

e For a prolate spheroid of eccentricity e magnetized along the axis of revolution

$$\lambda = 4\pi \frac{1 - e^2}{e^2}\left(\frac{1}{2e}\log\frac{1+e}{1-e} - 1\right)$$

and for an oblate spheroid

$$\lambda = 4\pi\left[\frac{1}{e^2} - \frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right]^*.$$

Thus r, which is $1 - \lambda/4\pi$, becomes for the former

$$\frac{1}{e^2}\left[1 - \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right]$$

and for the latter

$$\frac{1}{e^2}\left[e^2 - 1 + \frac{\sqrt{(1-e^2)}}{e}\sin^{-1}e\right].$$

* Maxwell, *loc. cit.*

These agree with Page's results, since $\cot^{-1}[\sqrt{(1-e^2)}/e] = \sin^{-1} e$.

For spheroids magnetized perpendicularly to the axis of revolution, if prolate

$$\lambda = \frac{2\pi}{e^2}\left[1 - \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right]$$

and if oblate

$$\lambda = 2\pi\frac{\sqrt{(1-e^2)}}{e^2}\left[\frac{1}{e}\sin^{-1} e - \sqrt{(1-e^2)}\right]^*$$

while it is easily shown that for elliptic cylinders magnetized transversely in the direction of the major axis

$$\lambda = 4\pi\frac{\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}},$$

and in the direction of the minor axis

$$\lambda = 4\pi\frac{1}{1+\sqrt{(1-e^2)}}.$$

The ratios of reduction of the magnetic field due to the surrounding medium may thus be readily calculated for these various cases.

In considering the effect of a surrounding medium upon the demagnetizing field and hence upon the magnetic induction in a spheroid so prolate that it approximates to a needle or so oblate that it approximates to a disc, Page has overlooked the fact that in the former case the demagnetizing force is negligible compared with the magnetization and therefore its diminution in a finite ratio does not sensibly increase the induction, while in the latter case although the change of the demagnetizing force is a very small fraction of its former value this change considerably affects the induction, which is the difference between $4\pi I$ and a nearly equal demagnetizing force.

It is obvious, in fact, that in the former case, since the field in the medium has $1/\mu$ of its value in a vacuum, the induction in the medium, and hence that in the needle, is unaltered, while in the latter case, since the field in the medium is the same as that in a vacuum, the induction in the medium, and hence that in the disc, is increased μ-fold.

The latter result can be verified as follows. Let the value of e for an approximate disc be $(1-\delta)$ where δ is small, then the corresponding value of λ becomes $4\pi - 2\sqrt{2}.\pi^2\sqrt{\delta}$ and that of α becomes $(\mu-1)2\sqrt{2}.\pi^2\sqrt{\delta}/4\pi$, if we neglect in each case higher powers of $\sqrt{\delta}$. The induction $(4\pi - \lambda)I$ when the disc is in a vacuum becomes $2\sqrt{2}.\pi^2 I\sqrt{\delta}$ and the induction $[4\pi - (1-\alpha)\lambda]I$, when the disc is in the medium, becomes $2\sqrt{2}.\pi^2\mu I\sqrt{\delta}$. δ

In the case of the disc, which is the limiting form of the uniformly magnetized oblate spheroid, the external magnetic force at its surface is constant from the centre to the very edge, as is evident from the fact that both the magnetization and the demagnetizing forces in it are constant. Its field therefore differs altogether from that of a simple magnetic shell, in which the force changes rapidly as the edge is approached and ultimately becomes very large. For the spheroidal disc, with its

* Maxwell, *loc. cit.*

constant finite external field, the induced magnetization in a surrounding medium will be everywhere finite, and as the distance separating the equal positive and negative surface distributions due to this magnetization is infinitesimal, being the thickness of the disc, it follows that the magnetic force in the medium due to the joint effect of these two distributions will vanish.

This argument accounts for the interesting result established by Page, that the permeability of the medium in which the disc is immersed has no effect at all on the field, but it casts grave doubt upon the legitimacy of the extension by mere analogy of this statement to the case of the simple magnetic shell, in which the magnetic force and consequent magnetization increase indefinitely in the neighbourhood of the edge. That the result still holds good, however, at least in a case readily amenable to calculation, namely that of a semi-infinite plane shell with a straight line as boundary, can be demonstrated as follows.

If a quantity of magnetism uniformly distributed along an infinite straight line
ρ with a density $+\rho$ per unit length is considered, the magnetic force at a point whose
r distance from the line is r will be a repulsion $2\rho/r$. Hence, if we next consider a semi-infinite plane with a straight boundary and a surface density of magnetism
σ, t $-\sigma$, and a point P whose perpendicular distance PN from the plane is t, where N
a is a point of the plane distant a from the edge, the component, perpendicular to the plane, of the magnetic force at P due to a strip parallel to the edge and at a
z distance z from N will be an attraction $2\sigma t\,dz/(z^2 + t^2)$, and therefore the total component magnetic force perpendicular to the plane is given by

$$2\sigma t\int_0^\infty \frac{dz}{z^2+t^2} + 2\sigma t\int_0^a \frac{dz}{z^2+t^2},$$

which reduces to

$$\sigma\left(\pi + 2\tan^{-1}\frac{a}{t}\right).$$

If now we consider a semi-infinite plane magnetic shell of thickness t and surface densities $\pm\sigma$, the component of the magnetic force perpendicular to the shell at a point just outside the positive surface and at a distance a from the edge will be the resultant of $2\pi\sigma$ outwards and $\sigma(\pi - 2\tan^{-1} a/t)$ inwards and thus become $\sigma(\pi - 2\tan^{-1} a/t)$ acting outwards.

When a is large compared with t we may take $\tan^{-1} a/t = \pi/2 - t/a$ and the expression for the component magnetic force assumes the well-known form $2\sigma t/a$.

If the shell is now surrounded by a magnetizable medium of constant susceptibility κ, the induced magnetism will produce additional surface densities over the positive and negative surfaces of the shell whose value will be

$$\mp\kappa\sigma(\pi - 2\tan^{-1} a/t)$$

if this distribution does not change the value of the magnetic field round the shell.

We proceed to inquire whether this condition is satisfied. If we have two infinite parallel straight lines at a distance t apart and with linear densities of magnetism $\pm\rho$ per unit length the magnetic force at a point equidistant from the lines and at a distance z from their plane is easily seen to be $2\rho t/(z^2 + t^2/4)$, the

direction of the force being the intersection of a plane parallel to that of the lines and a plane perpendicular to them. Hence if we consider a point Q within the shell midway between its surfaces and at a distance l from its edge and divide the shell into strips of width dx distant $l - x$ from the edge of the shell, the magnetic force at Q due to the surface distributions $\mp \kappa\sigma\,(\pi - 2\tan^{-1} a/t)$ will be

$$\int_{-\infty}^{l} dx\,\kappa\sigma \left[\pi - 2\tan^{-1}\frac{l-x}{t}\right]\frac{2t}{x^2 + t^2/4}.$$

We can evaluate this integral by using sufficiently approximate formulae to replace the factor $\pi - 2\tan^{-1}(l-x)/t$.

It may be shown that for values of $(l-x)/t$ between 0 and 1, and between 1 and 5, the respective empirical formulae $\pi t/(l + t - x)$ and $2{\cdot}15\,t/(l + 0{\cdot}37\,t - x)$ are adequate, while when $(l-x)/t$ is greater than 5 the approximate form $2t/(l-x)$ may be employed.

Inserting these, noting that

$$\int \frac{dx}{(b-x)(x^2+c^2)} = \frac{1}{b^2+c^2}\left[\log\frac{\sqrt{(x^2+c^2)}}{b-x} + \frac{b}{c}\tan^{-1}\frac{x}{c}\right]$$

and assuming that t is small compared with l we shall obtain for the magnetic force at Q

$$\frac{4\kappa\sigma t^2}{l^2}(2l\pi/t - \log t + \log l - \log 5 - 2) + \frac{4{\cdot}3\,\kappa\sigma t^2}{l^2}\log 3{\cdot}92 + \frac{2\kappa\pi\sigma t^2}{l^2}\log 2.$$

If we denote by ϕ the product σt, the strength of the shell, and assume it to be finite while t is infinitesimal, then since the limit of $t \log t$ is zero the above result reduces to $8\pi\kappa\phi/l$. As this is the product of 4π and $2\kappa\phi/l$, the value of the positive and negative surface densities in the immediate neighbourhood of Q due to the induced magnetization at the two sides of the shell, it follows that the magnetic force due to the joint effect of these surface densities at points in the medium just outside the shell will be zero, and therefore that it will be zero throughout the medium.

Thus the field round a semi-infinite thin simple plane magnetic shell with a straight boundary and of given strength is the same whether the shell is surrounded by a vacuum or by a magnetizable medium of constant permeability.

We have seen that in the case of any simple thin magnetic shell the induced magnetization very near its edge must bear the sole responsibility for a change in the field, if such a change exists, and we should therefore expect that curvature of the edge would be without influence if its radius were large compared with t. Thus it seems reasonable to conclude that the result obtained in the case of the semi-infinite plane shell is true for a thin simple shell of any shape.

If so, the result will also be true for any shell, not necessarily simple but everywhere thin and everywhere magnetized perpendicularly to its surface, for such a shell is resolvable into a suitable aggregate of simple shells.

538.214

A NEW METHOD OF DETERMINING THE MAGNETIC SUSCEPTIBILITIES OF GASES AND VAPOURS

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ABSTRACT. A new method of determining experimentally the magnetic susceptibilities of gases and vapours is described. Compensation is effected for the test bulbs, and the influence of surface condensation, if any, of gases and vapours can be allowed for and separately investigated. Electromagnetic retorsion is employed and the arrangement is rendered independent of small changes in the magnetizing current. The molar susceptibility of carbon dioxide was found to be $-(20{\cdot}79 \pm 0{\cdot}08) \cdot 10^{-6}$, showing good coincidence with the values of other investigators. The arrangement is being improved in several directions to give much greater accuracy. A systematic investigation of many gases and vapours will then be undertaken.

§ 1. INTRODUCTION

A large amount of work has been carried out* on the determination of the magnetic susceptibilities of gases, on account of the theoretical interest aroused by such measurements. The usual method is to suspend a small test body in a non-uniform field produced between the pole-pieces of an electromagnet. The suspension fibre is of phosphor-bronze or delicate quartz. The couple acting on the test body is determined either by noting the angle of twist at the lower end of the suspension, or the angle of retorsion which is necessary at the upper end to bring back the test body to its zero position. We are not aware of any other method employed to determine the angle of twist. The chief difficulties in such measurements are (i) the precise centring of the test body (see for example the elaborate precautions taken by Havens*; (ii) the strain effects in the retorsion method even in the most perfectly elastic fibres; (iii) the determination of the temperature in the neighbourhood of the test body; (iv) the great variations in the couple accompanying very small alterations in the magnetizing current; and (v) convection currents in the gases under examination, which disturb the positions of the light test bodies.

* For earlier work and detailed references see Havens, *Phys. Rev.* **43**, 992 (1933). Havens's results were published when we had already developed our method and taken down some measurements.

In the new method, our aim has been to avoid or correct for these various sources of error. Though our method is still being improved in several directions, we feel it advisable at this stage to publish the developments so far effected.

§ 2. EXPERIMENT

The magnetic field. The pole-pieces were cylindrical, having faces of diameter 8 cm. They were placed at a distance of 4 cm. from each other. To distort the magnetic field, two cylinders of good soft iron 7·5 cm. × 2 cm. were placed vertically as in figure 1 and held in position by a wooden block which was fixed tightly between the pole-pieces.

The test body. Thin sealed tubes A_1 and A_2, 12 mm. in diameter and 10 cm. long, were taken as the test bodies. Two other similar tubes C_1 and C_2 of exactly the same dimensions, but left open, acted as the compensating tubes. The four were bound together to form a rigid system and were carried on a central narrow tube of glass.

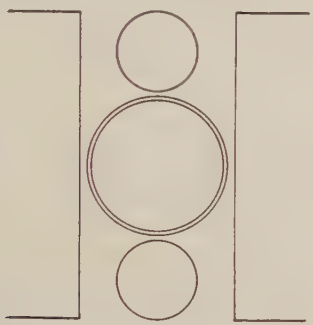

Figure 1.

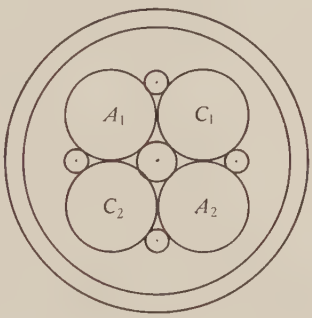

Figure 2.

The system weighed in all 34·71 gm. It was suspended by a fine platinum strip* of length 42 cm. and section 200μ × 50μ in the distorted magnetic field with A_1A_2 making 45° with the pole-pieces as shown in figure 2. The forces acting on the tubes A_1 and A_2 are approximately perpendicular to A_1A_2 and produce a twist in the suspension. A mirror attached to the central tube serves to measure the twist in conjunction with a lamp and scale.

Electromagnetic retorsion. A null method was effected by compensating the magnetic couple acting on the test object with an electromagnetic one. The suspension was connected to a coil comprising a single turn of platinum wire wound about the tubes. This coil terminated in a small thick platinum wire dipping vertically into some mercury below. The plane of the coil in the null position was arranged to be normal to the pole-faces. During the magnetic experiments, the current passing through it in the proper direction would give an opposing couple and this current could be arranged to bring back the coil, and hence the test body, to the initial position.

* We now find that a bifilar suspension is preferable and introduces fewer complications with increased sensitiveness.

The optical arrangement. The whole system was enclosed in a long wide glass tube through the top of which the suspension wire of platinum was sealed. A bulge blown carefully on the tube facilitated observation of the deflections of the mirror. A convex lens of long focal length was used to compensate for the refraction of this bulge. The image of the cross-wire was focused on a photographically reduced scale of $\frac{1}{10}$ mm. and observed with a low-power microscope.

§ 3. THE THEORY OF THE COMPENSATING METHOD

We can show by a rigorous calculation that if the glass of the test body is correctly compensated, the unbalanced couple on the system becomes

$$\iiint \frac{1}{2} \cdot \frac{dH^2}{d\theta} (k_a - k)\, dv_a \qquad \ldots\ldots(1)$$

k_a, k where k_a and k are the volume susceptibilities of the air in the test body and of the
H, θ, v_a surrounding gas in question, H is the magnetic field, θ the angle of twist, and dv_a an element of volume, the integral being extended over the volumes of the air in the test body. We may consider all the other effects to be balanced out.

The couple on the compensating coil of wire is given by

$$i \int Hr (\sin \theta)\, dl \qquad \ldots\ldots(2)$$

i, r where i is the compensating current in the coil of platinum wire, r is the radius of
l the coil of which dl is an element of length, and the line integral is taken along the circuit.

For a given geometric arrangement of iron, the field at any point is given by
f $H = f(k, \mu)\, I$, where $f(k, \mu)$ is some function depending upon the geometry of the
μ, I disposition and the permeability μ of the soft iron, and I is the current in the field coils. If the current is kept approximately constant, it can be assumed that this function at any point would remain constant. If this constant is taken as C, the expression (1) can be written as

$$I^2 \iiint \frac{1}{2} \cdot \frac{dC^2}{d\theta} (k_a - k)\, dv_a \qquad \ldots\ldots(3),$$

or

$$I^2 (k_a - k)\, K \qquad \ldots\ldots(4)$$

for any setting. The electromagnetic couple in (2) will become

$$iI \int C_1 r (\sin \theta).dl \qquad \ldots\ldots(5).$$

If the compensating current i is tapped as a definite fraction (r K) of the magnetic current I by a potential-divider, and is hence taken to be rI R, we obtain for the above

$$\frac{rI^2}{R} \int C_1 r (\sin \theta)\, dl \qquad \ldots\ldots(6).$$

p For the zero position of the coil this becomes prI^2l R where p is a constant and l the effective length. The condition for zero torsion is therefore

$$prI^2l/R = I^2 (k_a - k)\, K \qquad \ldots\ldots(7),$$

or

$$prl/R = (k_a - k)\, K \qquad \ldots\ldots(8).$$

We thus see that if R, the resistance in the compensating coil circuit, is kept constant, R

$$k_a - k \propto r \qquad \text{.....(9).}$$

Any small want of balance in the test body will merely introduce on the left a constant additive term which will become ineffective in our method of calculating the results.

Equation (8) is seen to be independent of any variations in the magnetizing current. It is to be noted that the compensating current is acted on by a magnetic field very close to the field which causes the deflection of the tubes, so that the compensation is local and hence nearly independent of external disturbances or small local displacements.

§ 4. ELECTRICAL CONNEXIONS

The electrical connexions are shown diagrammatically in figure 3. The magnetizing current of 8 A. was taken from a 120-V. battery B and measured with a care-

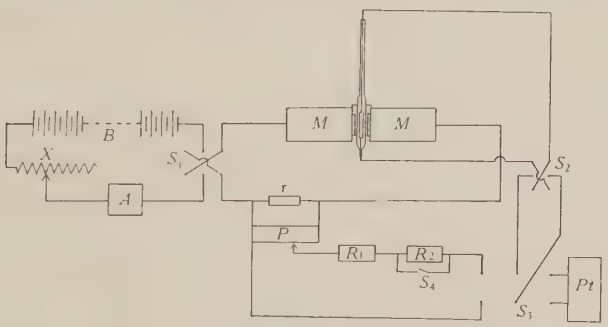

Figure 3.

fully calibrated ammeter A. S_1 is a commutator for reversing the current in the electromagnet M. The magnetizing current flows through a 0·1-ohm manganin resistance r of open type, which could be kept at a steady temperature by immersion in cold water. The potential at the ends of this low resistance served as the source of the compensation current. P is a potentiometer calibrated carefully to 0·1 per cent. It is of the Pye and Co.'s drum type with firm moving contact. R_1 is a resistance box and R_2 a single coil resistance of 100 ohms. The double-pole double-throw switch S_3 was made of six mercury cups in ebonite and amalgamated brass contact arms. The middle terminals were attached to the ends of the coil. The other two arms were connected to a platinum thermometer bridge Pt to indicate the mean temperature of the gas, the coil itself being used for the purpose. S_2 served for reversing the current in the coil for paramagnetic gases. S_4 was used in earlier experiments for short-circuiting the coil and served satisfactorily to control the oscillations of the system in the magnetic field. The null points were read with currents flowing in both directions. The residual field produced less than 0·1 per cent of the full-field deflection.

§ 5. READINGS WITH AIR AND CARBON DIOXIDE

By means of an elaborate system of glass connexions and a Cenco Hyvac pump, the experimental tube could be evacuated and flushed with pure dry air or gas repeatedly. The vacuum obtained was only a fraction of a millimetre. The pressures were read off on a mercury manometer open outside to the atmosphere, or at low pressures on a McLeod gauge. Air was passed over fresh soda lime and fused calcium chloride to free it from carbon dioxide and moisture. The carbon dioxide was generated by the action of pure dilute hydrochloric acid on marble, passed over glass wool dusted with pure sodium bicarbonate and then dried over fused calcium chloride. Check measurements with the platinum resistance thermometer bridge showed that the temperature inside did not vary by more than 0·05°.

Under proper conditions, the deflections for the gases were reproducible with a certainty of 1 in 1000 at pressures less than 50 cm. At higher pressures convection

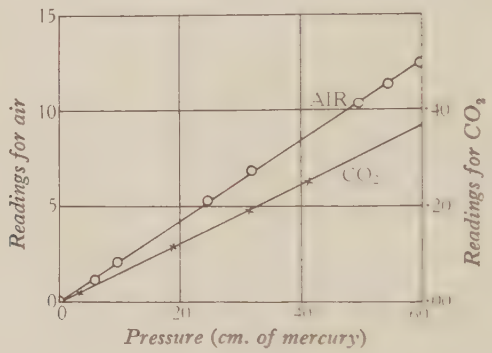

Figure 4.

currents gave small disturbances. We are now engaged in making some alterations in the apparatus which should bring down these disturbances to a minimum.

We have also recently introduced another coil of silver wire round the test body, so arranged that in the zero position the plane of this coil is nearly parallel to the pole-faces. This method of electromagnetic damping is very efficient and brings down the oscillations of the test body very rapidly.

In figure 4 are shown the curves between the pressure and the connected readings of the potentiometer. The corrections were arranged to be such that the deflection was zero when the pressure outside was the lowest obtainable. It will be seen that the points plot themselves on straight lines, thus presenting no Glaser anomalies. Havens obtained such anomalies even when the gases were dried on calcium chloride; but we have obtained no such variations. It is perhaps possible that continuous exposure of the chloride to the damp air increases the vapour pressure of the moisture in equilibrium with it.

Taking Sone's value of the volume susceptibility of air at normal temperature and pressure to be $0{\cdot}0308 \times 10^{-6}$, we obtain for the volume susceptibility of carbon

dioxide the value $-0{\cdot}000928 \times 10^{-6}$. The molecular susceptibility becomes therefore $20{\cdot}79 \times 10^{-6}$, which agrees well with the latest determination of

$$-(20{\cdot}88 \pm 0{\cdot}08) \times 10^{-6}$$

of Havens. It is believed that the error in our value is less than 0·4 per cent. Some preliminary experiments we have completed with benzene vapour indicate that this method is equally efficient for the study of the diamagnetic susceptibilities of vapours.

§6. ADVANTAGES OF THIS METHOD

We may here summarize the advantages of our method.

(i) Uncertainties in local fluctuations of densities due to temperature or pressure are rendered absent by making the test body sufficiently large in volume without increasing the weight. (ii) The magnetic field is definitely so distorted as to apply the maximum couple to the test body. (iii) The material of the container is adequately compensated. (iv) Proper facilities exist for separately studying and allowing for surface condensations. (v) Instead of mechanical retorsion, involving uncertainties even in the best experiments, we have here a method of electromagnetic retorsion applied to the body itself and not at the upper end. (vi) Since the current for the coil is tapped from the magnet circuit, small fluctuations of this current do not affect the equilibrium reading. (vii) The electromagnetic damping control on the test body makes easier the taking down of readings. (viii) A direct check on the inside temperature is possible by using the coil itself as a platinum resistance thermometer. (ix) Elaborate precautions for correct centring need not be taken, and the uncertainties on this account consequently disappear.

§7. ACKNOWLEDGMENT

We take this opportunity to thank the authorities of the Annamalai University for the award of a scholarship to one of us (G. Sivaramakrishnan) which has rendered this work possible.

DISCUSSION

Dr L. F. Bates. In Bitter's experiments on gases the test body used was a cylindrical glass vessel separated into four compartments or quadrants. Two opposite quadrants were evacuated and sealed, the others were open to the gas. Considerable skill in glass-blowing was obviously required in the manufacture of this test body. The authors have virtually used the same principle in their test body but they have shown that it can be constructed in a comparatively simple manner. It is interesting to note that apparently no surface-tension disturbances were observed at the platinum-mercury contact, and that the authors have designed a bifilar suspension which is more sensitive than a single torsion wire.

535.338.4:546.17

INTENSITY-DISTRIBUTIONS IN MOLECULAR SPECTRA: N_2 SECOND POSITIVE SYSTEM

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Read February 2, 1934.

ABSTRACT. Intensities of bands in the $C^3\Pi \rightarrow B^3\Pi$ system of N_2 under four different conditions of excitation have been measured by means of calibrated photographs of the spectrum; transition probabilities derived from these have been compared with the Condon parabola as obtained from Morse's and Rydberg's potential-energy functions; effective temperatures have been derived on the assumption of a Boltzmann distribution for vibrational energy; and an attempt has been made to explain the variation of intensities and transition probabilities under some of the conditions studied.

§ 1. INTRODUCTORY

The effect of temperature and excitation conditions on intensities within a band system has been but little investigated. The intensity changes due to pressure and influence of foreign gases have not been satisfactorily explained. In view of this, it is proposed to present in this paper accurate quantitative data on intensities of the second positive system of nitrogen produced under different experimental conditions and to study the results mainly from the standpoint of the Franck-Condon theory.

§ 2. EXPERIMENTAL

The intensity technique and photographic procedure adopted were precisely the same as those followed by Johnson and Tawde* on the C_2 Swan bands. On account of the extension of this system into ultra-violet region, a special ultra-violet-glass lamp capable of transmitting down to λ 2500 was procured from the General Electric Company Ltd. and calibrated with a quartz double monochromator in the usual way. The bands were photographed in two positions of the quartz prism on a Hilger E_1 instrument, the time for calibration marks and the spectrum being maintained equal by means of an automatic shutter. According to a suggestion made in the previously mentioned paper, slight fogging of plates preliminary to actual exposure was found to give satisfactory results.

The bands were excited in (i) the positive column of an ordinary end-on discharge tube; (ii) an uncondensed discharge in a mixture of 99 per cent of argon, 0·7 per cent of N_2 and 0·3 per cent of O_2; (iii) a high-frequency electrodeless discharge in air; and (iv) an uncondensed spark in air. It was also possible to produce

* *Proc. R.S.* A **137**, 575 (1932).

these bands in the arc of a quartz pointolite lamp, but experiments in this connexion had to be given up owing to accidental breakage of the lamp filament. During all exposures, the conditions of experiment such as pressure, temperature, current etc. were maintained strictly constant. The photographic intensities have been corrected for the energy-distribution (E_λ, λ) of the standard lamp.

§ 3. RESULTS OF OBSERVATIONS

The results of intensities expressed on the scale of 1000 for the (0, 0) band have been recorded in tables 1 to 4. Immediately below the numbers denoting intensities are given the I/ν^4 values of the bands, ν being calculated from the following equation for the N_2 ($C\,^3\Pi \rightarrow B\,^3\Pi$) system*:

$$= 29653{\cdot}1 + (2018{\cdot}66\,v' - 26{\cdot}047\,v'^2) - (1718{\cdot}4\,v'' - 14{\cdot}437\,v''^2).$$

The excitation function $\Sigma I/\nu^4$ for each initial vibrational level has been calculated by taking the sum of the I/ν^4 values of the bands in each v''-progression. This is denoted in the first column of each table. Taking the I/ν^4 value for a band in each v''-progression and dividing it by the corresponding excitation function for that progression, we get a number proportional to the transition probability. These numbers are entered in v', v'' schemes in figures 1 to 4.

With the aid of Morse's† potential-energy function, potential curves are drawn to scale in figure 5 for the $C\,^3\Pi$ and $B\,^3\Pi$ states of N_2, the constants used being as follows:

$$C\,^3\Pi: D_e' = 40087;\ a' = 3{\cdot}2973;\ r_e' = 1{\cdot}14\ \text{Å}.$$

$$B\,^3\Pi: D_e'' = 51995;\ a'' = 2{\cdot}4548;\ r_e'' = 1{\cdot}201\ \text{Å}.$$

From these curves the parabola of maximum transition probabilities has been derived by the graphical method of Condon. It has been denoted by the letter M in figures 1 to 4.

In place of that of Morse, Rydberg‡ has recently given a modified potential-energy function of the following form:

$$U(r) = D_e\,(ax + 1)\,e^{-ax},$$

where

$$a = \sqrt{\frac{\mu\,(0{\cdot}1735\,\omega_e)^2}{D_e}}$$

and

$$x = (r - r_e).$$

The constants used for N_2 in the above formula are:

$$D_e' = 40087;\ a' = 3{\cdot}3148;\ r' = 1{\cdot}14.$$

$$D_e'' = 51995;\ a'' = 2{\cdot}4666;\ r'' = 1{\cdot}201.$$

The resulting curves are plotted to scale in figure 6, except that the separation of the two scales used is arbitrary. The Condon parabola derived from these is shown by the letter R in figures 1 to 4.

* W. Jevons, *Report on Band Spectra*, p. 76 (Physical Society, 1932).
† *Phys. Rev.* **34**, 57 (1929).
‡ *Z. f. Physik.* **73**, 376 (1931).

Table 1. I and I/ν^4 values. Positive column in air

$\Sigma I/\nu^4$	v' \ v''	0	1	2	3	4	5	6	7	8	9	10	11
456·87	0	1000 129·33	968·7 158·75	578·8 121·47	155·1 42·14	12·3 4·39	1·43 0·68	0·16 0·11	—	—	—	—	—
297·40	1	841·5 83·9	121·8 15·16	268·4 42·03	453·5 90·49	214·5 54·89	25·2 8·39	4·84 2·12	0·70 0·41	—	—	—	—
192·24	2	202·1 15·87	451·5 43·69	102·4 12·30	134·7 20·24	247·1 46·87	163·8 39·63	30·9 9·66	7·89 3·21	1·43 0·77	—	—	—
139·60	3	365·7 23·09	293·4 22·55	201·1 18·96	198·7 23·11	3·22 0·46	85·5 16·22	89·9 20·67	35·7 10·49	8·21 3·12	1·88 0·93	—	—
64·28	4	—	—	217·1 16·43	105·1 9·67	151·4 17·11	4·76 0·66	12·0 2·09	35·2 7·72	25·98 7·23	6·51 2·32	1·88 0·87	0·29 0·18

Table 2. I and I/ν^4 values. Argon tube

$\Sigma I/\nu^4$	v' \ v''	0	1	2	3	4	5	6	7	8	9	10
301·42	0	1000 129·33	622·8 102·06	235·4 49·39	60·8 16·52	9·3 3·31	1·7 0·81	—	—	—	—	—
168·39	1	992·9 98·99	17·45 2·17	147·6 23·12	138·5 27·53	43·8 11·21	12·5 4·16	2·76 1·21	—	—	—	—
82·26	2	296·2 23·27	348·9 33·77	8·9 1·07	33·17 4·98	49·4 9·37	24·8 5·99	9·92 3·09	1·76 0·72	—	—	—
45·40	3	—	333·3 25·62	100·6 9·48	13·27 1·54	6·35 0·92	13·53 2·46	14·11 3·24	4·88 1·43	1·88 0·71	—	—
16·89	4	—	—	177·8 13·41	0·0 0·0	7·6 0·86	4·37 0·60	3·32 0·58	2·28 0·50	2·13 0·59	0·99 0·35	—

Table 3. I and I/ν^4 values. High-frequency electrodeless discharge

$\Sigma I/\nu^4$	v' \ v''	0	1	2	3	4	5	6	7	8	9	10
299·89	0	1000 129·33	769·4 126·07	145·4 30·51	46·22 12·55	4·02 1·43	—	—	—	—	—	—
171·68	1	968·9 96·57	35·9 4·46	225·6 35·21	87·64 17·47	54·8 14·02	11·86 3·95	—	—	—	—	—
111·45	2	669·9 52·58	328·1 31·73	19·48 2·33	32·63 4·90	52·14 9·88	31·3 7·57	7·91 2·46	—	—	—	—
61·56	3	—	506·9 38·96	147·9 13·94	27·5 3·19	4·21 0·59	8·65 1·56	10·31 2·36	3·27 0·96	—	—	—
9·39	4	—	—	34·18 2·57	39·85 3·66	11·21 1·26	0·0 0·0	6·86 1·15	1·99 0·44	1·13 0·30	—	—

Table 4. I and I/ν^4 values. Spark in air

$\Sigma I/\nu^4$	v' \ v''	0	1	2	3	4	5	6	7	8	9	10
279·82	0	1000 129·33	546 89·47	218·4 45·75	45·1 12·26	6·6 2·35	1·4 0·66	—	—	—	—	—
197·88	1	1047 104·38	18·93 2·35	207·8 32·55	189·5 37·71	52·99 13·56	19·27 6·41	2·10 0·92	—	—	—	—
164·76	2	712·6 56·0	660·2 63·88	29·56 3·55	28·29 4·25	101·9 19·33	40·28 9·75	21·55 6·72	3·14 1·28	—	—	—
105·33	3	—	552·3 42·45	413* 39·0	41·73 4·85	18·9 2·73	34·96 6·35	22·71 5·22	11·79 3·46	3·36 1·27	—	—
64·73	4	—	—	398·6 30·06	271·0* 25·0	27·51 3·11	0·0 0·0	10·63 1·35	12·10 2·65	5·89 1·64	2·59 0·92	—

* Determined by extrapolation.

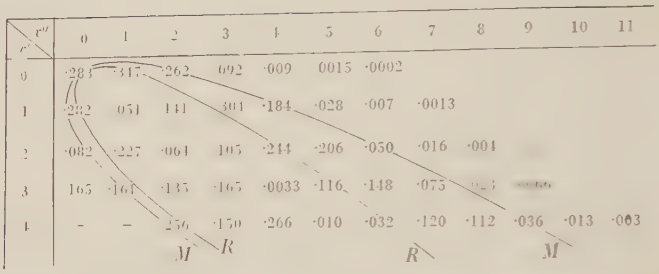

Figure 1. Transition-probabilities: positive column in air.

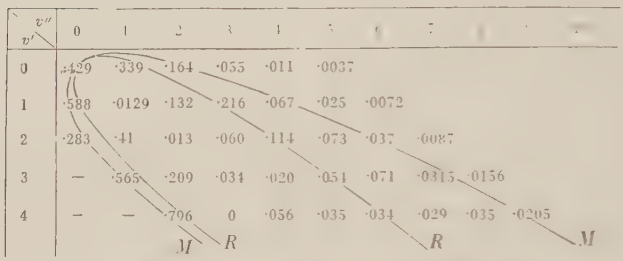

Figure 2. Transition-probabilities: argon tube.

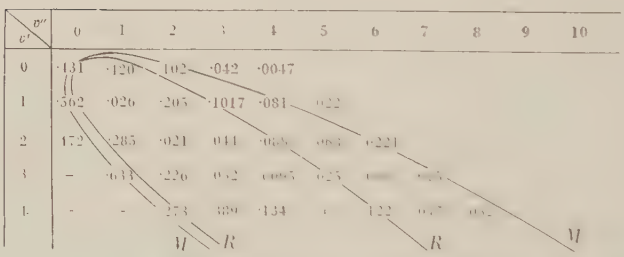

Figure 3. Transition-probabilities: high-frequency electrodeless discharge.

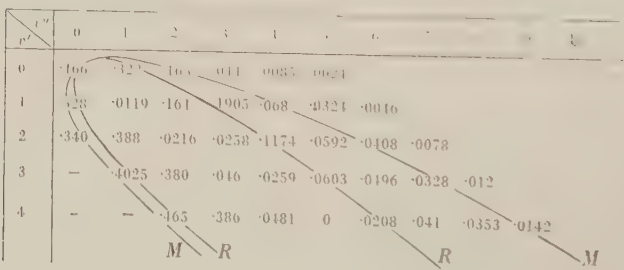

Figure 4. Transition-probabilities: spark in air.

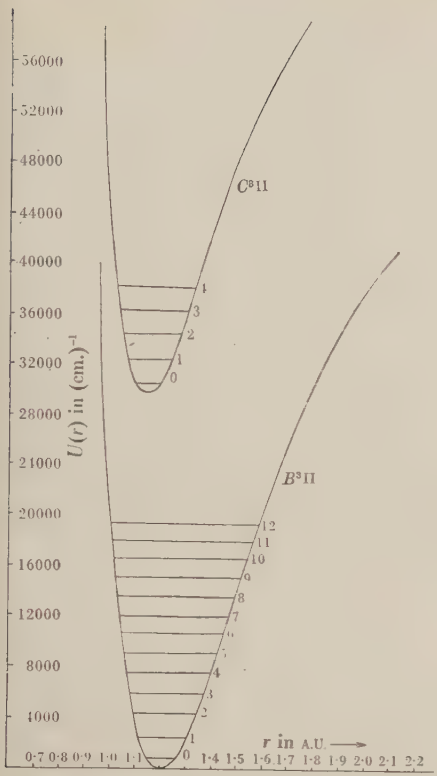

Figure 5. Potential curves for N_2 by Morse's function.

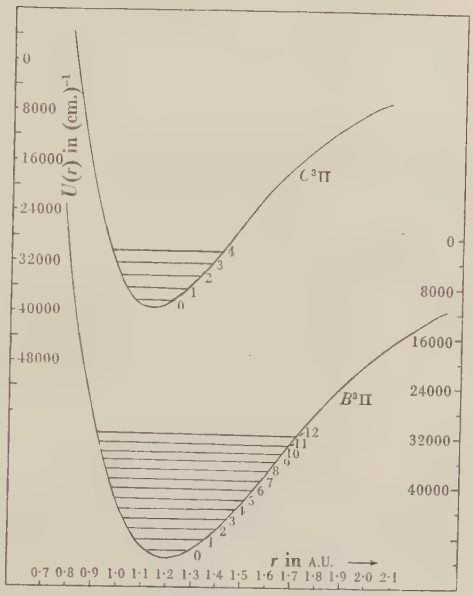

Figure 6. Potential curves for N_2 by Rydberg's function.

§ 4. EFFECTIVE TEMPERATURES

On the assumption that the number of molecules in each initial vibrational level are in thermal equilibrium at the temperature of the source, we should get

$$hc\,[\omega_e'\,(v'+\tfrac{1}{2}) - x_e'\omega_e'\,(v'+\tfrac{1}{2})^2] = kT\log_e(N_0/N).$$

Substituting the various constants for the upper $C\,^3\Pi$ state of N_2, and converting N_0 to $N_{\frac{1}{2}}$, we get

$$2044{\cdot}7\,v' - 26{\cdot}047\,v'\,(v'+1) = 1{\cdot}6083\,T\log_{10}(N_{\frac{1}{2}}/N_{v'+\frac{1}{2}}).$$

The numbers $N_{\frac{1}{2}}$ and $N_{(v'+\frac{1}{2})}$ are proportional to the excitation functions $\Sigma I/\nu^4$ in the respective initial vibrational levels. By plotting the logarithm of these ratios against the vibrational energy calculated from the left-hand side of the above equation, we get the absolute temperatures directly. The effective temperatures thus derived are:

(i) Positive column in air: 6730° K.

(ii) Argon-nitrogen mixture: 4670° K.

(iii) High-frequency electrodeless discharge: 5400° K.

(iv) Spark in air: 8420° K.

§ 5. CONCLUSIONS AND DISCUSSION

It appears from the results that the values of the relative intensities differ greatly from one excitation condition to the other. The intensities in table 1, relating to the positive column in air, are rather characteristic in the sense that, in contrast to what occurs in the case of the other conditions of excitation, the sequences corresponding to large change of quantum number are easily observable. On the other hand, the distribution in the high-frequency discharge is much nearer to that of excitation in high-pressure argon. It may be noted, however, that in all the cases studied there is a rise and fall of intensities along progressions such as is found for a normal band spectrum. The selective intensity of leading bands in sequences in the case of excitation in high-pressure argon makes it apparent that the effect observed is similar to one found by Johnson* in the first positive bands.

In connexion with the values of the intensities, a brief reference to the recent results of Okubo and Hamada† seems imperative. Those authors conclude that the lower the temperature of gas the better defined and sharper is the selective enhancement of bands with initial levels $v' = 0$ and $v'' = 1$. At higher temperatures the selective enhancement shifts to bands with quantum numbers greater than 0 and 1. In our results, only the bands (1, 0), (2, 0), (3, 1) are relatively more intense in high-pressure argon than in the positive column in air. This makes it apparent that it may not be the temperature effect that is alone responsible, as suggested by Okubo and Hamada, for the selective enhancement. Probably the change might be characteristic of the influence of inert gases. On the other hand, the examination of intensities in tables 3 and 4 shows that the bands (2, 0), (2, 1), (2, 2), (2, 4), (2, 5), (2, 6), (2, 7), (3, 1), (3, 3), (3, 4), (3, 5) and so on, belonging to v''-progressions with v' greater than 0 and 1 in the case of the spark in air, are relatively more enhanced than those in high-frequency electrodeless discharge. For bands with v' equal to 0 and 1, the enhancement is not at all pronounced. On the contrary, many of the bands belonging to these progressions are weaker in spark than in high-frequency discharge. These results are, therefore, in good agreement with Okubo and Hamada's conclusions referred to above, as molecules in a spark discharge are expected to be in a high-temperature atmosphere in comparison with high-frequency conditions.

The test of Condon's theory can be sought in figures 1 to 4. Here the results have been examined in the light of Morse's and Rydberg's potential-energy functions. The parabola as derived from Morse's formula appears wider than the locus of experimental figures, but it seems permissible to state that within the limits of experimental error the agreement with theory appears quite satisfactory. It is, however, interesting to note that a far better agreement results from the use of Rydberg's $U(r)$ function as may be seen from parabola R in figures 1 to 4.

According to Kaplan‡ the sudden curtailing of initial levels above $v' = 4$ in this

* *Phil. Mag.* **48**, 1069 (1924).
† *Astrophys. J.* **78**, 130 (1933).
‡ *Phys. Rev.* **37**, 1406 (1931).

system has been explained by assuming a third repulsive level due to the coming together of two metastable ^2D atoms, the transitions to which take place according to the Franck-Condon principle. This Heitler-London level cuts the potential curve for the C $^3\Pi$ state of N_2 in the neighbourhood of $v' = 5$. As a result of this the usual transitions to normal levels are quenched. If again, as has been suggested by Kaplan, such a level is sensitive to conditions of excitation, it should be possible to account for the variation of intensity within the progressions observed in tables 1 to 4 as a variation of intensity of interaction with normal levels.

In figure 7 is illustrated the effect of conditions of excitation on transition-probabilities. The transition-probabilities, derived from three cases studied here, are shown as a function of final vibrational quantum number in a v''-progression ($v' = 2$). The graph shows too high a probability for $v'' = 1$ in the case of excitation in high-pressure argon, in spite of the fact that a Heitler-London level due to the coming together of two ^4S atoms is supposed to cut the r_{max} part of

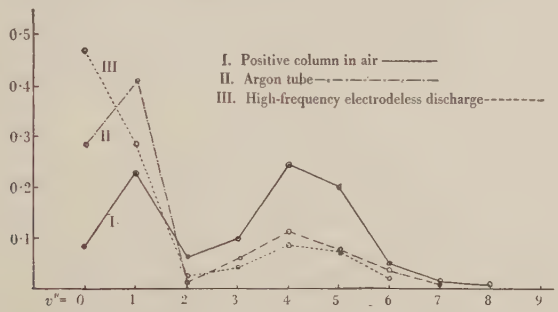

Figure 7. Transition-probabilities in v''-progression ($v' = 2$).

the normal potential curve for the B $^3\Pi$ state of N_2 in the neighbourhood of this vibrational level. Normally the r_{max} part of the Condon parabola as seen in figure 2 is considerably favoured relatively to r_{min} by reason of the assymmetry of the $U(r)$ curves. This shows that argon does not favour the transition to the Heitler-London level. If, for example, it were assumed that the degree to which the Heitler-London level is favoured depends upon the concentration of appropriate metastable atoms in the neighbourhood of radiating molecules, we might expect the effect illustrated in figure 7 in the case of argon. With this assumption, it is reasonable to suppose that the excess of argon atoms breaks up the fields produced by metastable nitrogen atoms in the neighbourhood of radiating molecules.

As regards the effective temperatures, almost a linear relation has been obtained between the vibrational energy and log (N_0/N) on the assumption of a Boltzmann distribution; but the temperatures derived are nowhere near the real temperatures of sources, nor do they bear a simple ratio to them. But owing to the presence of only a few initial levels in this system, we cannot draw any definite conclusions regarding the populations of various levels being in statistical equilibrium with their surroundings. However, the evidence available points to the conclusion that a vibrational distribution does not hold.

§ 6. ACKNOWLEDGMENTS

I am indebted to Sir Robert Robertson, F.R.S., of the Government Chemical Laboratories, London, for permission to use the monochromator, to Prof. A. Fowler, F.R.S., for permission to use the micro-photometer of the Imperial College, and to the General Electric Company, Wembley, for supplying the special ultra-violet-glass lamp. I desire also to express my gratitude to Dr R. C. Johnson, under whose direction the work was carried out; and to the High Commissioner for India, for the award of a research grant which enabled me to complete it.

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A NOTE ON THE TWISTING-FREQUENCY IN ETHYLENE

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ABSTRACT. From the experimental value for the fundamental twisting-frequency in ethylene, the magnitude of a certain carbon-carbon exchange integral J is determined as $0{\cdot}72 \pm 0{\cdot}10$ electron-volts. According to this result, the energy needed to twist one of the CH_2 groups through an angle $\pi/2$ with respect to the other about the C-C axis is $1{\cdot}0 \pm 0{\cdot}2$. This agrees well with the experimental value for the heat of activation of dimethyl maleate to dimethyl fumarate.

§ 1. INTRODUCTION

It has been known for some time from experimental evidence of several kinds[1] that the ethylene molecule is a plane one, but only recently was it possible to prove from quantum mechanics that this is the most stable configuration[2]. Intimately connected with the question of the stability of the molecule is the problem of calculating the amount of energy required to twist one of the CH_2 groups with respect to the other through an angle $\pi/2$ about the C-C axis. This quantity has been estimated very roughly at between $\frac{1}{2}$ and 1 electron-volt and it seemed impossible to get a more accurate value without tedious and difficult numerical calculation. We shall prove in the following note that a good estimate may be obtained from the observed fundamental twisting-frequency. In order to do this it is necessary to determine the force constant for small angles of twisting, a quantity which can be expressed very simply in terms of a certain fundamental exchange integral occurring in the theory of valency. Our problem resolves itself mainly into a determination of this integral.

Our calculations can be applied directly to ethane*. Here, however, there is very little resistance to twisting, as it is only the steric repulsions between hydrogen atoms in different CH_3 groups that hinder free rotation. Consequently, the fundamental twisting-frequency in ethane is very much less than in ethylene.

The ethylene molecule. On the assumption of perfect pairing of spins, the author[3] has shown that if one CH_2 group is twisted from the equilibrium position through

* The various energy levels for rotations of the ethane molecule about the C-C axis have been very roughly evaluated by Teller and Weigert, *Gott. Nach.* p. 218 (1933), in connexion with the specific heat. A more exact determination does not seem to affect their results on the specific heat very much.

ϕ an angle ϕ with respect to the other, the change in potential energy is given with sufficient accuracy* by

$$V(\phi) = -\tfrac{3}{4}J\cos 2\phi \qquad \text{.....(1),}$$

J where J is a certain exchange integral, the exact equation for which is not needed for our present purpose. Equation (1) is a good approximation because the conditions in ethylene are very favourable to the assumption of perfect pairing. Moreover, there is no need to worry about Coulomb terms, as one usually has to do in this sort of calculation, because these do not give any dependence on ϕ. Another pleasant feature of the equation is that one does not have to add terms to it to allow

W_s, W_p for the different ionization energies W_s and W_p of the $2s$ and the $2p$ electrons of carbon, as these too do not give any dependence on ϕ. This follows because the secular determinant for the ethylene molecule factorizes, and that part which contains ϕ does not contain W_s but only W_p. Moreover, the L-S coupling of the C electrons does not give a contribution to V as the $2p\pi$ electron on each carbon nucleus has its axis of symmetry perpendicular to the plane of the other three bonds for all values of ϕ.

The energy levels of a vibrating system in the field of potential (1) can be found quite simply by quantum mechanics; in fact, Koenig[3] has considered a

ν much more general case, where the potential field has ν maxima and minima in the range $0 \leqslant \phi \leqslant 2\pi$. The ethylene molecule is clearly a special case with ν equal to 2. When once we know the moment of inertia of the molecule about the C-C axis, we can apply the results of Koenig's paper to find the energy levels in terms of the constant J of equation (1). Now the fundamental twisting-frequency is known from experimental data, and it is not a difficult matter to find what value of J gives agreement with the experimental figure. The energy needed to twist the planes of the CH_2 groups perpendicular to each other is simply $\frac{3}{4}J$. Since there is only one parameter in the expression for the potential energy, it is reasonable to suppose that the process we have described for determining J will give a unique result, and we find that this is the case.

Consider now that the molecule is in one of these twisting states. Classically, of course, there is zero probability of one CH_2 group swinging through an angle π and commencing to oscillate about the new equilibrium position. If the kinetic energy at $\phi = 0$ is greater than $\frac{3}{4}J$, then the angular velocity varies in magnitude but never in sign. In the quantum mechanics, however, there is a finite probability that the molecule will change from oscillating about one equilibrium position to oscillating about the other, by penetrating the potential hill. As a consequence of this resonance the twisting-levels show a doubling. We have estimated the magnitude of this doubling and find that it is of the wrong order of magnitude to be observed.

* If ϕ is increased until it reaches the vicinity of $\pi/2$, the coupling of the $2p\pi$ (double bond) electrons to zero spin is broken down, and instead they are coupled to unit spin. Allowing for this, the difference of energy of plane and perpendicular ethylene is given by $\frac{1}{2}(3C' + C')$ instead of $\frac{3}{2}(C - C')$ given by equation (1). This latter therefore over-estimates the difference by $-2C'$, which is about 0·2 e.V.

§ 2. THE MATHEMATICAL CALCULATIONS

The wave equation to determine the twisting-levels is given by[4]

$$\frac{d^2\psi}{d\phi^2} + \frac{4\pi^2 I}{h^2}[W - V(\phi)]\psi = 0 \qquad \text{.....(2),} \qquad W$$

where $V(\phi)$ is given by equation (1), and I is the moment of inertia of one CH_2 group about the C-C axis. If we denote the HCH angle by $(\pi - 2\omega)$, the C-H distance by d and the mass of the hydrogen nucleus by M, then we have V, I ω d, M

$$I = 2Md^2 \cos^2\omega = 3{\cdot}31 \times 10^{-40} \qquad \text{.....(3),}$$

assuming for ω the calculated value 25° and for d the C-H distance in methane, $1{\cdot}10 \times 10^{-8}$ cm.

As an approximation to the solution of equation (2) we assume that there is considerable resistance to twisting, so that the amplitude of ϕ is small. It is therefore permissible to expand $\cos 2\phi$ in an ascending power series in ϕ, retaining at first only terms in ϕ^2. We have then the problem of the harmonic oscillator, and direct calculation with the value of I given in equation (3) shows that in order to fit the experimental value 750 cm^{-1} for the fundamental twisting-frequency[5], we must assume that

$$J = 0{\cdot}694 \text{ e.V.}$$

Since J comes out so much bigger than the fundamental itself we have justified our expansion in ϕ, and at the same time have obtained an approximate solution of equation (2). It should be explained that it was by no means obvious without examination that there was not another value for J which would give the same fundamental. A very shallow trough, for example, might have had only two vibrational levels, differing by just the fundamental. This is the possibility which our work eliminates, and we show in fact that the harmonic-oscillator approximation is quite a good one.

In order to obtain more accurate values for J and W we use the process developed by Koenig. The expansion in ϕ is made, and the effect of the various terms is calculated by straightforward perturbation theory. This method is delightfully simple and the convergence for the lowest energy levels is rapid. For those levels approaching the rotator the convergence is not so good, and one would probably have to use the implicit method described by Koenig. However, we are not interested in these levels and therefore do not worry about them. Modifying Koenig's perturbation calculations to suit our example, we find that the energy W_n of the nth twisting-state in ethylene is given by W_n n

$$W_n = \frac{h}{\pi}\left[\frac{3J}{2I}\right]^{\frac{1}{2}}[(n + \tfrac{1}{2}) - (2n^2 + n + 1)/16k - (2n^3 + 3n^2 + 3n + 1)/128k^2 \ldots]$$

$$\text{.....(4),}$$

where $kh = \pi[\frac{3}{2}JI]^{\frac{1}{2}}$. Using our value $J = 0{\cdot}694$, we obtain $k = 11{\cdot}3$. Substituting back in equation (4) we find that it is necessary to adjust J to 0·718 e.V. in order to preserve the fundamental twisting-frequency at 750 cm^{-1} k

To illustrate the points we have just explained, we give in the table the energy levels of the harmonic oscillator W_h with our final value for J of 0·718 e.V., and with them the accurate values of W_a calculated from the field of potential (1). It will be noticed that the higher levels deviate considerably from the harmonic-oscillator approximation; in other words, there is a considerable convergence factor. Since any twisting-frequency is inactive, the twisting-levels have to be obtained experimentally from combination bands, and with these the intensity falls off rapidly with the order. There is little hope at the moment, therefore, of comparing our convergence factor with the experimental value.

n	0	1	2	3	4
W_h	381	1144	1907	2670	3433
W_a	377	1127	1859	2572	3265

§ 3. COMPARISON WITH PREVIOUS WORK

We have seen that in order to get the right value for the fundamental twisting-frequency in ethylene it is necessary to assume that the exchange integral J has the value 0·72 e.V. This gives about 1 e.V. for the energy required to twist one CH_2 group perpendicular to the other, an estimate which should be accurate to within 20 per cent. The value given by Pauling and Wheland [7] for the exchange integral J in benzene is 1·50 - 0·10 e.V., obtained from thermo-chemical data. Values were assigned to various chemical bonds and were adjusted to give the correct free energies of various compounds in which there is no possibility of resonance between different structures. Using these values for structures in which there is resonance, the difference between the estimated and observed free energies must be due to the resonance itself. The calculated effect of the resonance is simply a numerical multiple of J, and thus J is determined. The present author is a little sceptical of the accuracy which is claimed for this method. One would not expect the integral J to be exactly the same in ethylene as it is in benzene, but the two should differ by only a few per cent. The difference between our estimate and Pauling and Wheland's is too large to be accounted for in this way.

The only attempt at a numerical calculation of J is that of Bartlett [7], and there is a little difficulty in reconciling his figures with ours. It is not possible to get agreement between the values of Bartlett and those of Pauling and Wheland. If one assumes that the effective nuclear charge of the carbon atom towards the $2p\pi$ electron of the double bond is 2·5 and that the internuclear distance is 1·2 A., Bartlett's value for J is 0·74 e.V. This agrees well with our figure but has been obtained only by assuming a rather large effective nuclear charge.

There is one other way of obtaining a check on the value of J, and that is from the observed heat of activation of compounds derived from ethylene. If one of the H atoms in each CH_2 group is replaced by a ($COOCH_3$) group, there are two possible forms for the resulting molecule, and it is possible to distinguish between

them because they have different chemical properties. The kinetics of the reaction has been studied by Nelles and Kistiakowsky[8], who found that the heat of activation of one form to the other is 26·5 kcal./mol. (1·15 e.V.). They concluded from chemical evidence that the activation energy represents almost exclusively the straining of the double bond on a 90° rotation of one relative to the other of the two atomic groups of the molecule. Although the bonds in this example are not quite the same as those in ethylene, one would expect the dependence of the energy W on ϕ to be very similar in the two cases. This is borne out by the good agreement between the above value 1·15 e.V. of Nelles and Kistiakowsky and our value 1·0 ± 0·2 e.V., obtained from the fundamental twisting-frequency.

In conclusion, we consider the possible error which exists in our estimate of J. First of all there is some uncertainty in the exact value for the fundamental, but this should not be more than a few wave-numbers, unless indeed the 750 $cm.^{-1}$ is not a twisting-frequency at all. This possibility seems to be ruled out fairly conclusively by the observed character of the infra-red absorption bands. A much more likely source of error is in the value we have used for the moment of inertia about the C-C axis. The C-H distances and the angle ω are not known very accurately and there may easily be an error of 5 per cent in I. This gives a possible error of about 10 per cent in J.

§ 4. ACKNOWLEDGMENT

The author wishes to thank Drs Sutherland and Melville for helpful discussion.

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ANOMALOUS CHANGES IN TEMPERATURE DUE TO THERMIONIC EMISSION IN THE FILAMENTS OF VALVES

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ABSTRACT. A triode valve should show, for a given filament current, a steady filament-temperature, lower when the anode is at a positive than when it is at a negative or zero potential. It has been found that in some valves the steady filament-temperature is lower when the anode is positive, as would be expected, but in other valves it is higher. It is shown that this anomalous increase in temperature is due to radiation from the anode and is larger for valves having a high anode dissipation and an anode which closely surrounds the filament. It is shown that when a correction for this effect has been applied the work function can be approximately calculated from measurements made on an ordinary valve.

Richardson's equation for the thermionic saturation current is $i = aT^{\frac{1}{2}}e^{-\phi/kT}$,

a, T, ϕ a being a constant, T the absolute temperature, ϕ the thermionic work

k function, supposed independent of T, and k Boltzmann's constant. The average energy of thermal agitation of the electrons is $2kT$ so that they carry away per second a total of u ergs, where

$$u = \frac{i}{e}(\phi + 2kT),$$

e e coulombs being the electronic charge.

V On writing $\phi = Ve$ we have

$$u = i\left(V + \frac{2k}{e}T\right) = iw,$$

w where w is the loss of energy per ampere per second due to the thermionic emission.

Any valve filament will thus be cooled by the emission of electricity, and there will be a change in its resistance. From the measured change in resistance one can find the change in temperature.

Experiments on this effect have been made by various workers[1,...,8]. The results obtained in the present investigation are apparently partially opposed to those obtained by several previous observers, especially Lester[4] and Davisson and Germer[6,7]. It must be particularly noted, however, that these observers worked with carefully designed *diodes* constructed specially for investigation purposes. The observations made were clearly of an accurate nature, but the difference in the present case is that ordinary well-known commercial makes of *triode* have been

submitted to test. Certain aspects of anomalous behaviour in such commercial triodes are what the present paper records.

Tieri and Ricca[9] pointed out in 1928 that the emission entails a variation in the effective heating-current, and that this provides a means of finding the intrinsic potential of some metals. They also were working with triodes.

On change of the anode potential from a large negative to a positive value there will be a decrease in filament temperature due wholly to the emission, provided that the temperature of the surroundings does not change and that the supply of heat is unaltered. The electron current decreases the effective heating-current so that there will be a drop in temperature due to this cause. But unless the anode is specially cooled the filament will be in an enclosure of increasing average temperature. The extra heat received from the enclosure will diminish the cooling, and when equilibrium is reached may be enough to mask it completely. Whether or not this will happen depends on the shape of the anode and its rise in temperature.

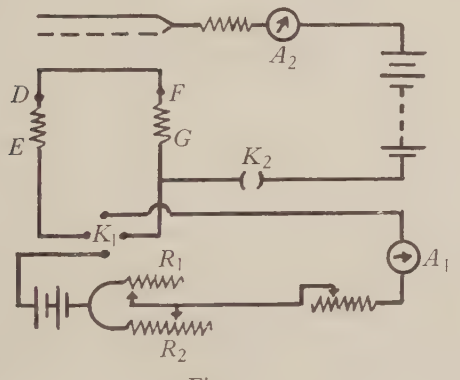

Figure 1.

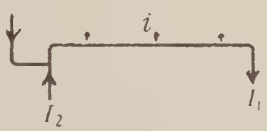

Figure 2.

The experimental arrangements are indicated in figure 1. A_1 and A_2 are ammeters showing respectively the approximate magnitudes of the heating and anode currents. The rheostats R_1 and R_2 in parallel regulate the filament current. DE and FG are two 1-ohm standard resistance coils, of which the potential terminals can be connected to a potentiometer through a suitable commutator. The keys K_1 and K_2 serve to open or reverse the battery connexions to the filament and anode. The potential-differences V_1 and V_2 between the terminals of the resistance coils and V_3 between the filament pins were measured in the usual way with a potentiometer. Accumulators were used as sources of potential. In all cases the grid and anode were directly connected.

V_1, V_2
V_3

On reference to figure 2 it is clear that I_1 and I_2, the currents in the negative and positive leads to the filament, are unequal, the difference $(I_1 - I_2)$ or i being the thermionic current. As i is measured directly by the milliammeter A_2 a leak would be shown at once by a discrepancy between the readings. Assuming the emission to be constant along the length of the filament the effective heating-current will be very nearly $\frac{1}{2}(I_1 + I_2)$, the mean of the values at the ends.

I_1, I_2
i

The potential-difference V_3 between the filament terminals being measured, *R* we have for the resistance R of the filament

$$R = \frac{2V_3}{I_1 + I_2} - s.$$

The pins, supports, and leads for the filaments in the valves used in this work are not surrounded by the anode; consequently their resistance will not vary much *s* with its temperature. s, which is the resistance of these conductors, will therefore be nearly constant, and being small may be neglected.

If curves be drawn with the filament-resistances as ordinates and heating-currents as abscissae, for the cases where the anode is (*a*) negative and (*b*) positive, their general form will be that shown in figures 4 and 5. It appears that the filament-temperature is decreased by the emission; but in some cases, as in valves like Mullard P.M. 252 for example, the curves have the general shape of those in figure 3, indicating an increase of temperature when the anode is positive. The

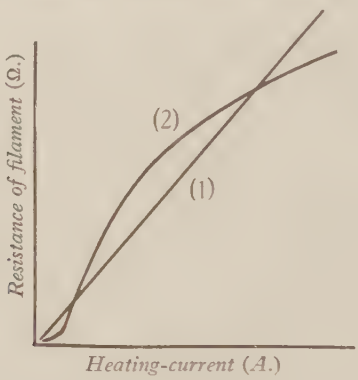

Figure 3. (1) H.t. off; (2) h.t. on.

resistance of the filament varies with the time during which the anode current has been flowing, and so does the increase in temperature in these cases. Four valves of this type gave nearly the same results. Other valves such as Osram L.S. 5 which have a larger anode surface per watt dissipated, and ones like R. 5 V. with a smaller total anode dissipation, do not show this heating.

That this reversal of the usual effect is due to heat received from the anode is made clear if the course of an experiment is so ordered that the anode remains at a nearly constant temperature. This may be done by closing the anode circuit and immediately thereafter the galvanometer circuit of the potentiometer connected across the filament pins; if balance is not obtained both circuits are opened, an interval is allowed for cooling, and the potentiometer is readjusted. The process is continued until balance is obtained. Alternatively the anode circuit may be kept normally closed and measurements made when it is momentarily opened. The observations recorded were made by the first method.

A Mullard P.M. 252 valve was examined in this way; the curves have the form shown in figures 4 and 5. There is now no range over which heating takes place.

From the curves showing the relations between i and R, figures 4 and 5, we can calculate approximately the work function ϕ for the material of the filament. Suppose that a line parallel to the I axis is drawn to cut the curves. The points of intersection have for abscissae I' and I'', the values of the heating-current required to bring the filament to the same resistance in the two cases. From the curve

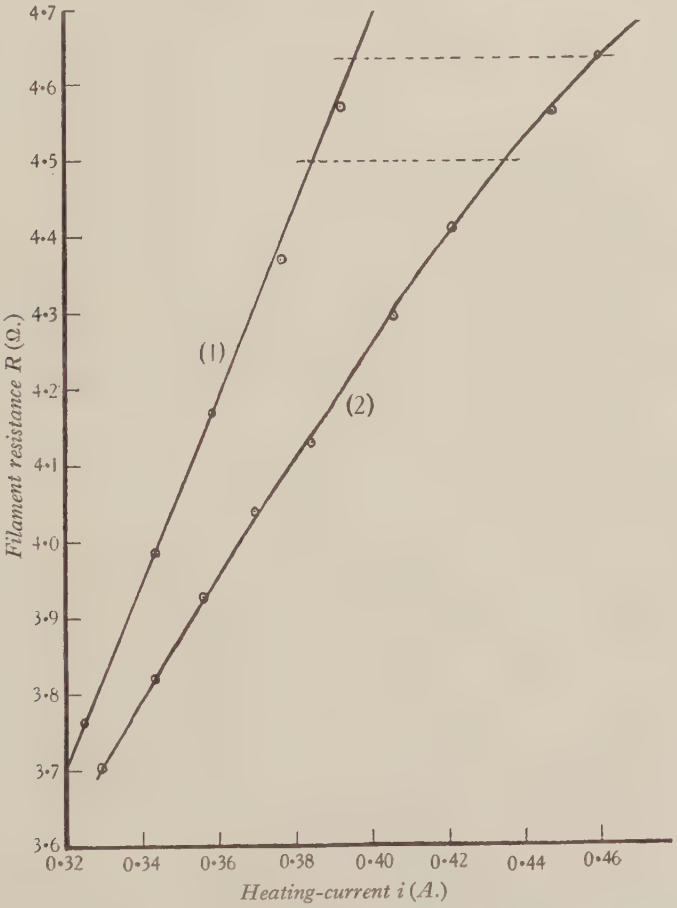

Figure 4. Mullard P.M. 252. H.t. to negative end of filament. (1) H.t. off; (2) h.t. on.

showing the relation between I and i, figure 6, we find the thermionic emission. Since the filament has the same temperature in both cases the losses by conduction, radiation, etc. are equal, and the energy u carried away by the electrons is given by

$$u = (I'^2 - I''^2)\, R,$$

while $w = u/i$. Since the number of electrons leaving per second is $i/1{\cdot}5 \times 10^{-19}$ the loss L in ergs per electron is given by

$$L = 1{\cdot}5 \times 10^{-12}\, w \text{ ergs.}$$

A summary of results on four kinds of valve is shown in the table.

Table

Type of valve	i (mA.)	L (ergs $\times 10^{-12}$)	End of filament to which h.t. is connected
P.M. 252	124·5	3·28	+
	95·7	3·04	
	120·6	3·38	−
	95·7	3·03	
L.S. 5	125·0	7·25	+
	107·2	6·76	
	120·3	6·70	−
	137·2	6·85	
R. 5 V. (an old specimen)	74·1	10·72	+
	57·0	10·01	
	81·6	10·20	−
	62·0	9·90	
V.T. 10	77·4	7·55	+
	62·7	7·32	
	84·1	6·69	−
	65·7	6·61	

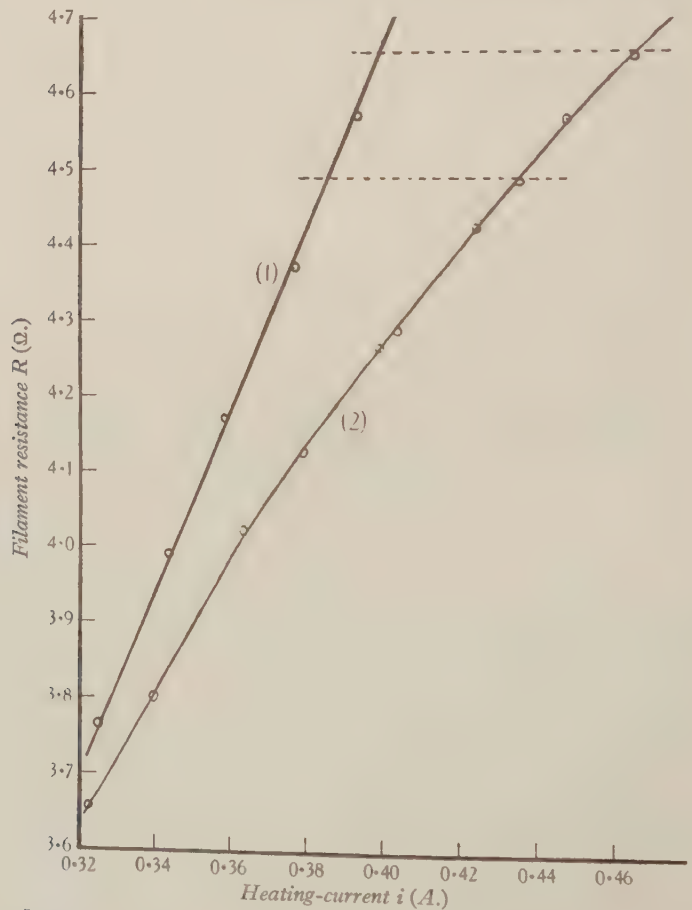

Figure 5. Mullard P.M. 252. H.t. to positive end of filament. (1) H.t. off; (2) h.t. on.

To sum up we may say that: (1) In an ordinary small valve, the cooling of the filament due to its thermionic emission may be masked by radiation from the anode. (2) Whether the effect takes place or not depends on the construction of the valve and on the working-temperature of the anode. Power valves may show a considerable rise in temperature. (3) The work function can be calculated where suitable corrections are made, though with considerable error.

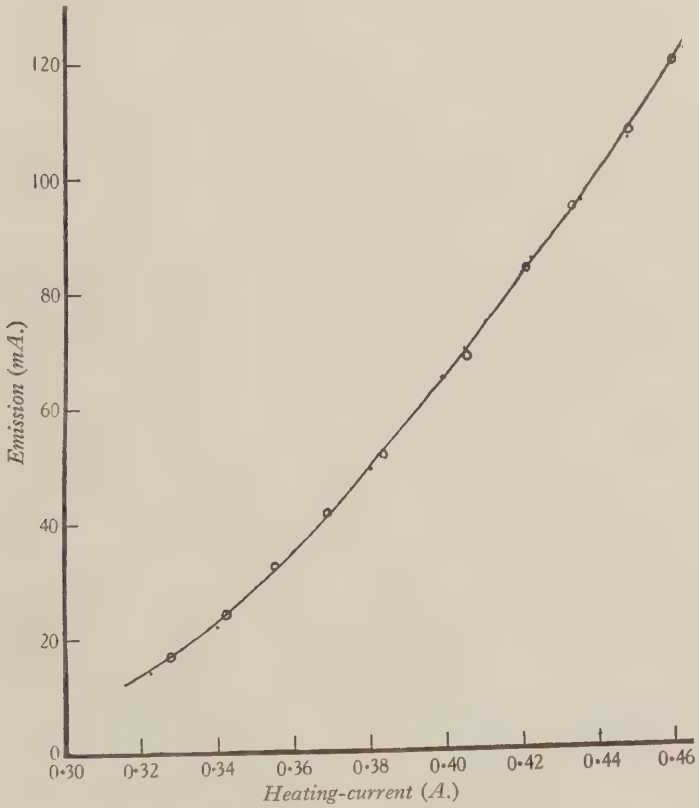

Figure 6. Mullard P.M. 252. • H.t. to positive end of filament.
○ H.t. to negative end of filament.

In conclusion I should like to express my thanks to Prof. T. L. Rosgyll Ayres for his encouragement and for placing the resources of the laboratory at my disposal.

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CHANGE OF VARIABLES IN LAPLACE'S AND OTHER SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Transformations of variables are expressed as matrix products, the effect of transposition being particularly considered, and the results applied to the transformation of the general second-order differential expression.

Changes in the independent variables in Laplace's and other equations occurring in mathematical physics are usually considered separately for each equation, and are often carried out in a somewhat clumsy fashion. A convenient general form in which to express the result of changing from a set of variables $\alpha, \beta, \gamma, \delta, \ldots$ to an equivalent set $a, b, c, d, \ldots$ is obtainable if the equation is expressed as a matrix product. Let J denote the Jacobian $\dfrac{\partial(a, b, c, d, \ldots)}{\partial(\alpha, \beta, \gamma, \delta, \ldots)}$, j the corresponding matrix

$$\begin{pmatrix} a_\alpha & a_\beta & a_\gamma & \ldots \\ b_\alpha & b_\beta & b_\gamma & \ldots \\ c_\alpha & c_\beta & c_\gamma & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

(where suffixes indicate differentiation), and j' the matrix conjugate to j formed by turning the first, second ... horizontal row into the first, second ... vertical column or *vice versa*. Also let Δ and D respectively denote the operators

$$\left(\frac{\partial}{\partial\alpha} \quad \frac{\partial}{\partial\beta} \quad \frac{\partial}{\partial\gamma} \ldots\right) \quad \text{and} \quad \left(\frac{\partial}{\partial a} \quad \frac{\partial}{\partial b} \quad \frac{\partial}{\partial c} \ldots\right)$$

regarded as linear matrices, and Δ' and D' the corresponding transposed vertical operators. Then if ϕ is the dependent variable, ψ a given function of $\alpha, \beta, \gamma, \ldots$ and Θ a square matrix, of order not exceeding the number of independent variables, whose elements are either constants or functions of $\alpha, \beta, \gamma, \ldots$, *the equation*

$$\Delta\Theta\Delta'\phi = \psi$$

becomes on change of variables

$$JDJ^{-1}j\Theta j'D'\phi = \psi,$$

where, in accordance with the usual convention, the differential operators act on all factors written on their right. This formula is interesting as showing that the simplification of all equations of this type depends upon finding a j which reduces $j\Theta j'$ to a canonical form.

For readers who are not accustomed to the use of matrices it may be helpful to illustrate by an example the connexion between an expression $\Delta\Theta\Delta'\phi$ and the ordinary form. Let us suppose there are three independent variables α, β, γ, and that Θ denotes the matrix

$$\begin{pmatrix} l & m & n \\ p & q & t \\ u & v & w \end{pmatrix},$$

where the elements l, m, ... w may be either constants or functions of α, β, γ. Then, in accordance with the fundamental matrix convention that in multiplication a horizontal row of a left-hand factor is associated term by term with a vertical row of a right-hand factor, $\Delta\Theta\Delta'\phi$ denotes

$$\frac{\partial}{\partial\alpha}\left(l\frac{\partial\phi}{\partial\alpha}+m\frac{\partial\phi}{\partial\beta}+n\frac{\partial\phi}{\partial\gamma}\right)+\frac{\partial}{\partial\beta}\left(p\frac{\partial\phi}{\partial\alpha}+q\frac{\partial\phi}{\partial\beta}+t\frac{\partial\phi}{\partial\gamma}\right)+\frac{\partial}{\partial\gamma}\left(u\frac{\partial\phi}{\partial\alpha}+v\frac{\partial\phi}{\partial\beta}+w\frac{\partial\phi}{\partial\gamma}\right).$$

It often happens that a relation expressed by a differential equation involves the differential coefficients in a way which naturally finds its expression in a form of this kind. The matrix Θ can then be written down immediately. One of the merits of the transformation formula to be established is that when the variables are changed the new equation is obtained directly in this convenient form.

When the coefficients in the equation are constants the matrix can always be written down immediately. Thus the equation

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}+\lambda\left(\frac{\partial^2U}{\partial x\partial z}+\frac{\partial^2U}{\partial y\partial z}\right)=V$$

is formally the same as

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}\Theta\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}U=V,$$

where

$$\Theta=\begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

provided λ is a constant or a function of z only. If λ is a constant we may write Θ in other forms, for example the symmetrical form

$$\begin{pmatrix} 1 & 0 & \frac{1}{2}\lambda \\ 0 & 1 & \frac{1}{2}\lambda \\ \frac{1}{2}\lambda & \frac{1}{2}\lambda & 0 \end{pmatrix}$$

by making use of the relations

$$\frac{\partial^2U}{\partial x\partial z}=\frac{\partial^2U}{\partial z\partial x},\quad \frac{\partial^2U}{\partial y\partial z}=\frac{\partial^2U}{\partial z\partial y}.$$

When the elements of Θ are not constants we shall usually have first order differential

coefficients of the dependent variable in the equation, and there may also be a term involving the dependent variable undifferentiated. We may bring all these terms into the standard form we have assumed by introducing a dummy unit variable. For example if the coefficients $a, b, c, \ldots$ are arbitrary functions of x and y, the expression

$$av_{xx} + 2hv_{xy} + bv_{yy} + 2gv_x + 2fv_y + cv$$

may be written

$$\left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right)\begin{pmatrix} a & h & g' \\ h & b & f' \\ g' & f' & c' \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} vz^2,$$

where after differentiation z is replaced by unity, and

$$4f' = 2f - h_x - b_y$$
$$4g' = 2g - a_x - h_y$$
$$4c' = 2c - 2g_x - 2f_y + a_{xx} + 2h_{xy} + b_{yy}.$$

When there are n independent variables the total number of coefficients in the linear second order expression is $\frac{1}{2}(n-1)(n-2)$ and this is exactly the number of independent elements in the symmetrical square matrix of order $n-1$. It follows that this matrix form for the equation is always possible. By the use of an unsymmetrical matrix a dummy variable can often be avoided without difficulty. Alternatively we may use the transformation formula to be obtained for the second order terms only, and deal with the remaining terms by the simple first order relations.

The most familiar of all transformations to the physicist is perhaps that from rectangular cartesian to polar coordinates. The fundamental equations are

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

The values of j and J may be derived from these indirectly by making use of the fact that j is the inverse of the matrix i employed later. A direct determination follows from the equivalent equations

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\{(x^2 + y^2)^{\frac{1}{2}} z^{-1}\}, \quad \phi = \tan^{-1}(yx^{-1}).$$

The expressions reached are

$$j = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r^{-1}\cos\theta\cos\phi & r^{-1}\cos\theta\sin\phi & -r^{-1}\sin\theta \\ -r^{-1}\operatorname{cosec}\theta\sin\phi & r^{-1}\operatorname{cosec}\theta\cos\phi & 0 \end{pmatrix}$$

and

$$J^{-1} = r^2\sin\theta.$$

If this transformation is to be applied to the equation which, written in full, is

$$\frac{\partial}{\partial x}\left(l\frac{\partial\kappa}{\partial x} + m\frac{\partial\kappa}{\partial y} + n\frac{\partial\kappa}{\partial z}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial\kappa}{\partial x} + q\frac{\partial\kappa}{\partial y} + t\frac{\partial\kappa}{\partial z}\right) + \frac{\partial}{\partial z}\left(u\frac{\partial\kappa}{\partial x} + v\frac{\partial\kappa}{\partial y} + w\frac{\partial\kappa}{\partial z}\right) = k,$$

and which, as we have already seen, corresponds to

$$\Theta = \begin{pmatrix} l & m & n \\ p & q & t \\ u & v & w \end{pmatrix},$$

the result in polar coordinates, written at length, is

$$\frac{\partial}{\partial r}\left(L\frac{\partial \kappa}{\partial r} + M\frac{\partial \kappa}{\partial \theta} + N\frac{\partial \kappa}{\partial \phi}\right) + \frac{\partial}{\partial \theta}\left(P\frac{\partial \kappa}{\partial r} + Q\frac{\partial \kappa}{\partial \theta} + T\frac{\partial \kappa}{\partial \phi}\right)$$
$$+ \frac{\partial}{\partial \phi}\left(U\frac{\partial \kappa}{\partial r} + V\frac{\partial \kappa}{\partial \theta} + W\frac{\partial \kappa}{\partial \phi}\right) = kr^2 \sin\theta,$$

where the new coefficients are given by the matrix equation

$$\begin{pmatrix} L & M & N \\ P & Q & T \\ U & V & W \end{pmatrix} = r^2 \sin\theta \,.\, j\Theta j' .$$

As a simple example take

$$l = q = w = \rho, \quad -m = p = \nu,$$
$$n = -u = \mu, \quad -t = v = \lambda.$$

On multiplying together the three matrices we easily find

$$L = \rho r^2 \sin\theta, \quad Q = \rho \sin\theta, \quad W = \rho \operatorname{cosec}\theta,$$
$$M = -P = r\sin\theta\,(\lambda \sin\phi - \mu\cos\phi),$$
$$N = -U = r\,(\lambda\cos\theta\cos\phi + \mu\cos\theta\sin\phi - \nu\sin\theta),$$
$$V = -T = \lambda\sin\theta\cos\phi + \mu\sin\theta\sin\phi + \nu\cos\theta.$$

We may now turn to the theorem itself. The only feature of the proof which is not quite direct is that which results in the introduction of the factors J and J^{-1} on inverting the order of some of the matrices. It does not involve any actual difficulty in a direct proof, but we shall make use here of a theorem of Jacobi's. This states that if A, B, C, ... are the cofactors of the elements in order of any row of the Jacobian

$$\begin{vmatrix} \alpha_a & \alpha_b & \alpha_c & \cdots \\ \beta_a & \beta_b & \beta_c & \cdots \\ \gamma_a & \gamma_b & \gamma_c & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

then

$$\frac{\partial A}{\partial a} + \frac{\partial B}{\partial b} + \frac{\partial C}{\partial c} + \ldots = 0.$$

The sum on the left clearly involves only products of first differential coefficients with one second differential factor in each term. Taking the top row for illustration, the factors involving double differentiation with respect to a and b can only arise

from the first two terms, and their sum is obviously

$$\begin{vmatrix} 1 & . & . & . & \dots \\ . & \beta_{ab} & \beta_c & \beta_d & \dots \\ . & \gamma_{ab} & \gamma_c & \gamma_d & \dots \\ . & \delta_{ab} & \delta_c & \delta_d & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} + \begin{vmatrix} . & 1 & . & . & \dots \\ \beta_{ab} & . & \beta_c & \beta_d & \dots \\ \gamma_{ab} & . & \gamma_c & \gamma_d & \dots \\ \delta_{ab} & . & \delta_c & \delta_d & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

which vanishes. Similarly all the other terms vanish.

Now let i denote the matrix

$$\begin{pmatrix} \alpha_a & \alpha_b & \alpha_c & \dots \\ \beta_a & \beta_b & \beta_c & \dots \\ \gamma_a & \gamma_b & \gamma_c & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

i' the transposed matrix, and I the corresponding determinant. Then, using also Δ, Δ', D, D', j, j' and J in the sense explained in enunciating the theorem, the ordinary equations of transformation for differential coefficients are

$$\Delta' = j' D' \text{ and } D' = i' \Delta'.$$

These involve $ij = ji = 1$, and $IJ = 1$. If we wished to transpose these equations as they stand we should have to adopt some such form as

$$\Delta = Dj_0, \quad D = \Delta i_0,$$

where the zero suffixes indicate that the expressions to which they are attached are not to be subject to the operators which immediately precede them. Such restriction is most inconvenient, and we proceed to remove it.

Let $A_\pi, B_\pi, C_\pi \dots$ be the cofactors of the row $\pi_a, \pi_b, \pi_c \dots$ of I. Then by Jacobi's theorem, whatever ϕ may be

$$\frac{\partial A_\pi \phi}{\partial a} + \frac{\partial B_\pi \phi}{\partial b} + \frac{\partial C_\pi \phi}{\partial c} + \dots = A_\pi \frac{\partial \phi}{\partial a} + B_\pi \frac{\partial \phi}{\partial b} + C_\pi \frac{\partial \phi}{\partial c} + \dots.$$

But since $ij = 1$

$$A_\pi = Ia_\pi, \quad B_\pi = Ib_\pi,$$

and so on, and the equation above is therefore the same as

$$DI \begin{pmatrix} a_\pi \\ b_\pi \\ c_\pi \\ \dots \end{pmatrix} \phi = (A_\pi \quad B_\pi \quad C_\pi \quad \dots)\, D'\phi$$
$$= (A_\pi \quad B_\pi \quad C_\pi \quad \dots)\, i'\Delta'\phi$$
$$= I \frac{\partial \phi}{\partial \pi}$$

by a fundamental property of the elements of a determinant and their cofactors. Putting $\alpha, \beta, \gamma, \dots$ in turn for π we have

$$DIj\phi = I\Delta\phi,$$

or since $IJ = 1$

$$\Delta\phi = JDJ^{-1} j\phi.$$

As ϕ is arbitrary we may replace it by $\Theta\Delta'\phi$, which, as we have already seen, is equal to $\Theta j' D'\phi$; the general second order transformation formula

$$\Delta\Theta\Delta'\phi = JDJ^{-1} j\Theta j' D'\phi$$

then follows.

To illustrate the application of this formula take $\Theta = 1$; on the left $\Delta\Delta'\phi$ represents the general Laplacian expression. Its form will be unaltered on changing the variables if, and only if, $jj' = 1$, i.e. if j is an orthogonal matrix, or if a = const., b = const., etc., intersect mutually at right angles. More generally by substituting any scalar function f for Θ we see that the same conclusion holds for the equation

$$\frac{\partial}{\partial\alpha}\left(f\frac{\partial\phi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(f\frac{\partial\phi}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(f\frac{\partial\phi}{\partial\gamma}\right) + \ldots = \psi.$$

550.838

AN INSTRUMENT FOR ELECTRICAL PROSPECTING BY THE INDUCTIVE METHOD

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ABSTRACT. This paper deals with an examination of the conditions in the Bieler-Watson method of geophysical surveying, and demonstrates that, in general, the horizontal field is not in quadrature with the vertical field. An instrument has been designed which will allow the horizontal field to be compared completely with the vertical field, the important feature of the instrument being that the horizontal components in phase and in quadrature with the vertical field are obtained directly from the instrument readings. The apparatus has been tested on elliptically polarized fields and has given satisfactory results.

§ 1. INTRODUCTION

THE work described is the outcome of an attempt to improve the usefulness of the Bieler-Watson method of electrical prospecting, by modifying it in such a way that the associated magnetic fields may be measured more precisely and without neglecting components which in fact exist. The Bieler-Watson method was used with some success during the Imperial Geophysical Experimental Survey in Australia in 1928-30, but experiences in the field showed in what respects it was lacking.

§ 2. THE INDUCTIVE METHODS OF PROSPECTING

In the inductive method of geophysical prospecting, an alternating magnetic field is applied to the ground under examination, by means of a large loop of wire conveying alternating current and laid on the ground-surface. The magnetic field induces eddy currents in the neighbouring rocks and consequently the resultant magnetic field at any point is made up of two parts, the primary applied field and the secondary field due to the eddy currents. In general the resistivities of the rocks are large, so that the eddy currents and the secondary field are small. If, however, a good conductor, such as a sulphide-ore body, is in the vicinity, the secondary field may attain values of the same order as the primary field. By making measurements of the magnetic field at the earth's surface the position of the good conductor may be inferred from the large values of the secondary field.

If it is assumed that the surrounding rocks are non-conductors the secondary field will be due to the good conductor only. The e.m.f. producing the currents is proportional to the rate of change of magnetic flux through the body, and will differ in phase by 90° from the primary magnetic field. Further, if the eddy-current paths have negligible reactance the currents themselves, and hence the secondary

field, will be in quadrature with the primary field. Since the two fields have different directions the resultant field to be measured is elliptically polarized. At points on the ground-surface the primary field is vertical and, if the measurements are made within the loop at a sufficient distance from the cable, it is constant in amplitude to 2 per cent over a large area. If the vertical component of the secondary field is small compared with the primary field, the vertical field in the loop may be regarded as a magnetic field of standard amplitude and phase. The horizontal component of the secondary field is measured in terms of the vertical field, and with the assumptions made these two fields are in quadrature.

The Bieler-Watson* apparatus used for this determination consists of two coils at right angles so that there is no mutual interaction between them, one coil being placed in a vertical plane and the other in a horizontal plane. The e.m.f.s induced

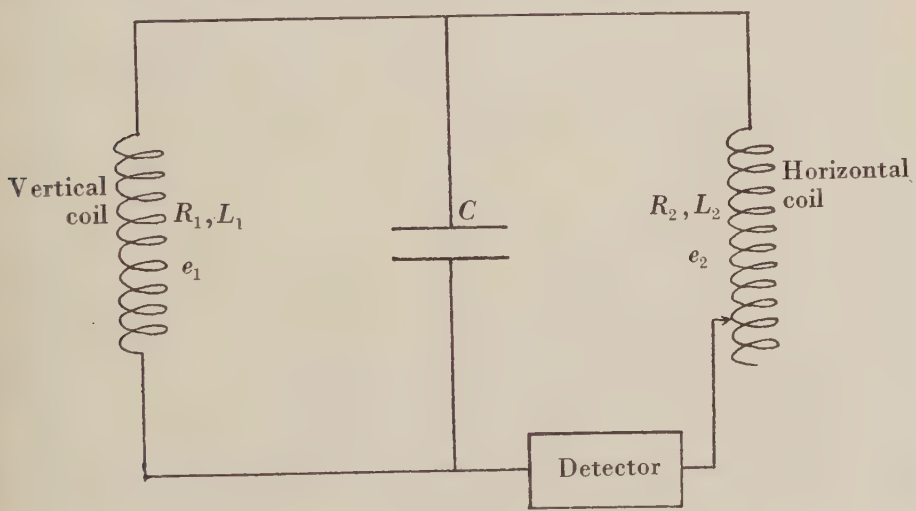


Figure 1.

R_1, L_1, R_2, L_2

in the coils by the magnetic fields are in quadrature, but if the vertical coil is tuned with a condenser the e.m.f. across the condenser will be in phase with the e.m.f. in the horizontal coil, and by varying the number of turns in the latter the system can be balanced, figure 1. Let H and V be the amplitudes of the horizontal and vertical fields, and e_1 and e_2 the amplitudes of the corresponding e.m.f.s in the two coils. Then

H, V e_1, e_2

$$e_1/e_2 = A_V N_V . H/A_H N_H . V,$$

where A_V, A_H are the effective areas of each turn and N_V, N_H the numbers of turns of the vertical and horizontal coils respectively. With the notation of figure 1 the balance condition for the circuit is given by

A_V, A_H, N_V, N_H

ω

$$e_1/e_2 = -j\omega C R_1$$

or

$$H/V = -j\omega C R_1 . A_H N_H / A_V N_V = -jkN_H \qquad \ldots\ldots(1),$$

* Broughton, Edge and Laby, *Principles and Practice of Geophysical Prospecting* (Camb. Univ. Press, 1931).

where k is a constant of the instrument. Hence the number of turns of the horizontal coil required for balance is a measure of the ratio V/H. Thus by measuring N_H at various parts of the field the variation of the horizontal component of the secondary field may be obtained and the position of the good conductor inferred. Equation (1) can only be satisfied if H and V differ in phase by 90°, and for any other phase relation balance cannot be obtained. In practice it is found that minima only are obtained, and in many cases the minima are so broad that the readings are useless. The occurrence of minima rather than true balance points has been attributed to harmonics, in spite of the fact that tuned amplifiers were used as detectors. Efforts were made to improve the sharpness of balance by rocking the system about a horizontal axis, in which case the greatest intensity-variation will be obtained on balance if the residual note is due to harmonics. This method is open to suspicion, as the balance condition may not correspond to the vertical position of the tuned coil.

There are many reasons, both practical and theoretical, why the simple assumption concerning the quadrature relation between the fluxes threading the two coils is not justified. A small tilt of the tuned coil, or a deviation of the primary field from the vertical due to local topography, will produce an in-phase e.m.f. in the vertical coil. At the same time the vertical component of the secondary field will produce a quadrature component in the horizontal coil. On theoretical grounds it is to be expected that the primary magnetic field will be appreciably absorbed as it passes down through the country rock to the good conductor, and in general this is accompanied by a phase-change. Hence the flux through the ore body is no longer in phase with the primary field, and for the same reason the secondary field at the surface is not in phase with the eddy currents producing it. The skin effect in the ore bodies is usually pronounced on account of their dimensions and low resistivity; consequently the phase of the eddy currents will vary from point to point, again influencing the phase of the secondary field. In many cases the magnetic field due to a current passed conductively through the ground is examined, and here the resultant field is even more complicated in structure.

It is apparent that a more suitable instrument is necessary for this type of prospecting, which will allow the phase as well as the amplitude of the horizontal component to be determined. The object of the present investigation is to examine methods which would allow this to be accomplished and, if possible, to design a direct-reading instrument which would allow the in-phase and quadrature components of the horizontal field to be measured in terms of the vertical field without further calculation.

§ 3. CIRCUITS INVOLVING PHASE-MEASUREMENTS

By introducing a small resistance in the condenser arm of the original double coil circuit, small variations of the horizontal field from quadrature may be taken into account. With the notation of figure 2

$$\left.\begin{aligned}(R_1 + r + L_1 j\omega - j/\omega C).u - (s - j/\omega C).v - e_1 = 0\\ -(s - j/\omega C).u + (R_2 + s + G + L_2 j\omega - j/\omega C).v - e_2 = 0\end{aligned}\right\} \quad \ldots\ldots(2),$$

whence, if $L_1 w = 1/wC$, the balance condition becomes

$$e_1/e_2 = -(R_1 + r)/(s - j/\omega C).$$

From this equation, both the in-phase and the quadrature component of e_1 and hence of H may be determined. If the balance condition be written in the form

$$e_2 = -u(s - j/\omega C),$$

it is seen that phase angles in the quadrant 0 to $-\pi/2$ may be obtained, while, by reversing the leads to the horizontal coil, phases in the quadrant $\pi/2$ to π may

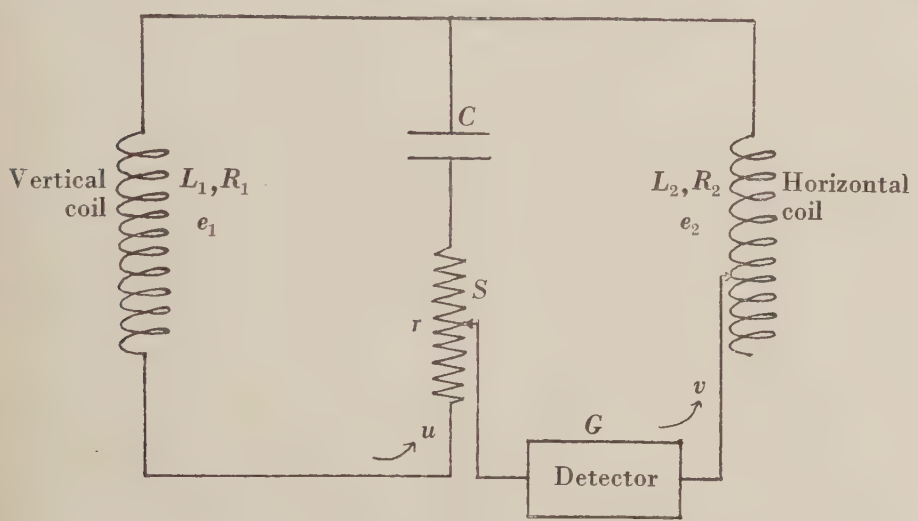


Figure 2.

be measured. For small deviations from quadrature the instrument is direct-reading, for

$$|e_1/e_2| \doteq \omega C(R_1 + r)$$

and the angle of deviation δ, assumed small, is given by

$$\delta = \tan^{-1}(-\omega C s) \propto s.$$

By replacing the condenser by a variable mutual inductance, with both search coils fixed, a system which allows any phase angle to be measured is obtained. The balance is secured by varying the mutual inductance and the tapping of the resistance, figure 3. If L_2, R_2 are the total inductance and resistance in the current AB, and L_1, R_1 the corresponding values for the vertical coil C and the secondary winding of the mutual inductance, the equations for the currents in the two nets are

$$\left.\begin{aligned}(R_2 + L_2 j\omega).u - (s + Mj\omega).v - e_2 &= 0\\ -(s + Mj\omega).u + (R_1 + s + L_1 j\omega).v - e_1 &= 0\end{aligned}\right\} \quad \ldots\ldots(3)$$

and the balance condition is

$$e_1/e_2 = -(s + Mj\omega)/(R_2 + L_2 j\omega)$$

or

$$e_1 = -u(s + Mj\omega).$$

Since the signs of both u and M may be changed, the instrument allows all phase angles to be measured since any combination of signs in the formula

$$e_1 = -(s \pm Mj\omega).u$$

ω_0 may be obtained. For any frequency $\omega_0/2\pi$ an instrument of the type may be made direct reading if the mutual inductance is calibrated to give $M\omega_0$ in ohms. If in equation (3) $R_2 = L_2\omega_0$ the balance condition assumes the form

$$\frac{e_1}{e_2} = -\frac{1}{R_2}\left\{\frac{s}{1+j} + j\frac{M\omega_0}{1+j}\right\}$$

$$= -\frac{1}{2R_2}\{(M\omega_0 + s) + j(M\omega_0 - s)\} \qquad \ldots\ldots(4).$$

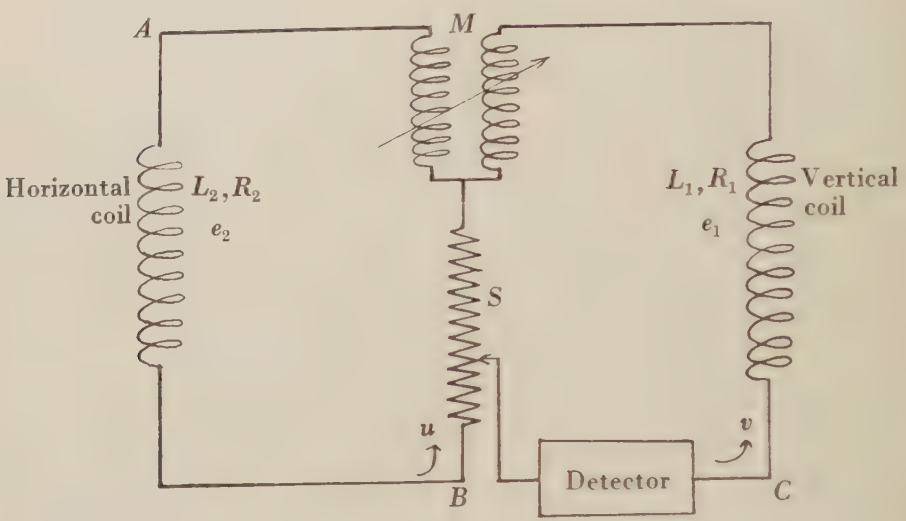


Figure 3.

Hence addition and subtraction of the readings gives the in-phase and quadrature components immediately. For any other frequency, if $R_2 = L_2\omega$,

$$\frac{e_1}{e_2} = -\frac{1}{2R_2}\left\{\left(M\omega_0.\frac{\omega}{\omega_0} + s\right) + j\left(M\omega_0.\frac{\omega}{\omega_0} - s\right)\right\}$$

and a simple factor serves to convert both readings to the same scale.

One advantage of this last circuit is that the balance is independent of the constants of the vertical coil, and hence, by replacing this coil by others having different numbers of turns the range may be extended indefinitely with only a small change in sensitivity if a valve amplifier (acting as a perfect voltmeter) is used as a detector, since the same fraction of e_2 is always available for balancing purposes.

§ 4. THE DIRECT-READING INSTRUMENT

An improvement will be effected if two scales can be obtained whose readings give the two components directly without further calculation. This can be done by means of the circuit shown in figure 4. The two branches in parallel across AB act as a non-inductive resistance R and, further, the currents in the two branches differ in phase by $\pi/2$ if $L = CR^2$, a condition independent of frequency. If at the normal working frequency the additional condition $L\omega = 1/\omega C = R$ is imposed,

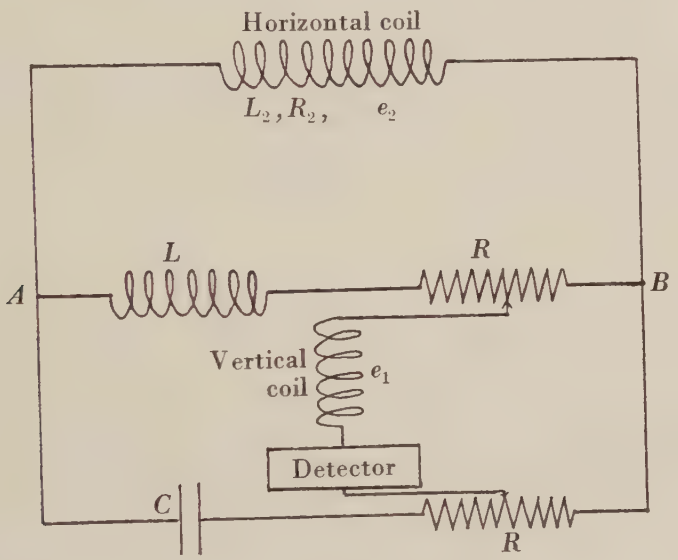


Figure 4.

the two currents will be equal in magnitude and differ in phase from the main current u in the circuit by $\pm \pi/4$. This current is given by

$$u = e_2/\{R + R_2 + L_2 j\omega\}$$

and if R_2 can be adjusted so that $L_2\omega = R + R_2$, u and e_2 will differ in phase by $\pi/4$. Of the currents in the two branches, one will be in phase and one in quadrature with e_2; consequently when the e.m.f. in the vertical coil is balanced, the readings on the two potentiometer scales give direct measures of the in-phase and quadrature components of e_1. The circuit shown in figure 4 will only allow two quadrants of the (phase, amplitude) diagram to be used, but by making the two potentiometer branches symmetrical this difficulty may be overcome.

§ 5. THEORY OF THE INSTRUMENT

If the e.m.f. in the horizontal coil is e_2 and the corresponding e.m.f. set up in the vertical coil is e_1, then, with the notation of figure 5, the currents in the various networks are related by the equations

$$\left.\begin{aligned} &\{R + R_2 + (L + L_2)\,j\omega\}.u - (R + Lj\omega).v - (\tfrac{1}{2}R + r + \tfrac{1}{2}Lj\omega).g - e_2 = 0 \\ &-(\tfrac{1}{2}R - r + \tfrac{1}{2}Lj\omega).u + (R - r - s + \tfrac{1}{2}Lj\omega - \tfrac{1}{2}j/\omega C).v - Xg - e_1 = 0 \\ &-(\tfrac{1}{2}R + r + \tfrac{1}{2}Lj\omega).u + (R + r + s + \tfrac{1}{2}Lj\omega - \tfrac{1}{2}j/\omega C).v \\ &\qquad + (R + r + s + X + \tfrac{1}{2}Lj\omega - \tfrac{1}{2}j/\omega C).g + e_1 = 0 \end{aligned}\right\} \quad \ldots(5),$$

r, s where r and s are the tappings on the potentiometers measured from the central

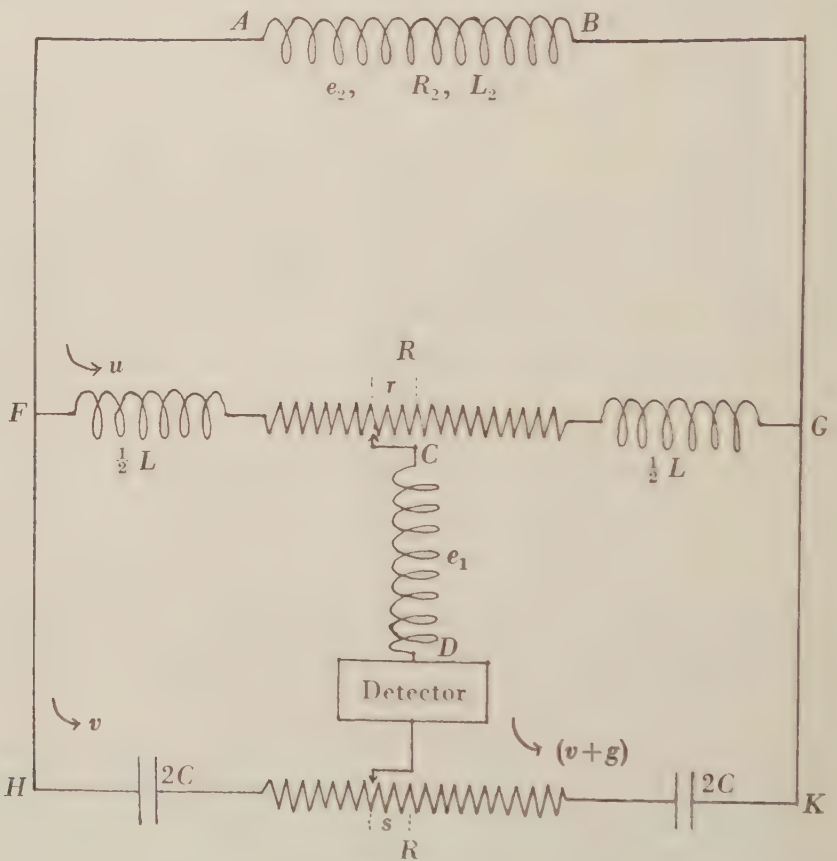


Figure 5.

X point and X the impedance of the detector arm. From these, the condition that
g the current g through the detector shall be zero can be written in the form

$$e_1\,[\{R + R_2 + (L + L_2)\,j\omega\}\,(2R + Lj\omega - j/\omega C) - (R + Lj\omega)^2] = e_2\,\{r\,(R - j/\omega C) - s\,(R + Lj\omega)\}.$$

If now $L = CR^2$ then $R + Lj\omega = j\omega CR\,(R - j/\omega C)$ and hence

$$e_1\{j\omega CR\,(R_2 + L_2 j\omega) + R + R_2 + (L + L_2)\,j\omega\} = e_2\{r - j\omega CR.s\} \quad \ldots\ldots(6).$$

The readings r and s will be direct measures of the in-phase and quadrature components of e_1 if the coefficient of e_1 in the above expression is real or imaginary. The condition that the coefficient shall be real cannot be realized, but if

$$R_2 = R\,(L_2 C\omega^2 - 1)$$

it becomes imaginary and

$$\frac{e_1}{e_2} = -\frac{\omega CR}{\omega\,(CRR_2 + L + L_2)}.s - j\frac{1}{\omega\,(CRR_2 + L + L_2)}.r \quad \ldots\ldots(7).$$

The two readings do not give the two components on the same scale, but this is not important from the practical standpoint, the interpretation of one component being used to supplement the other. The other two scales will be identical if $\omega CR = 1$ in which case

$$L\omega = 1/\omega C = R,$$

$$L_2\omega = R_2 + R$$

and

$$\frac{e_1}{e_2} = -\frac{1}{2(R + R_2)}.(s + jr) \quad \ldots\ldots(8).$$

For any other frequency the components may be reduced to the same scale by multiplying the in-phase component by ω/ω_0 where $\omega_0/2\pi$ is the frequency at which the scales are equal and $\omega/2\pi$ the applied frequency.

§ 6. SOURCES OF ERROR

(i) *Error due to tilt of search coils.* Since in practice the primary vertical field is much larger than the horizontal field, any slight inclination of the vertical coil will result in a relatively large in-phase flux through the vertical coil. A rotation of the coil system through 180° about a vertical axis will reverse the e.m.f. due to the horizontal field but not that due to the component of the primary field. Consequently the effect of the horizontal field alone may be determined from the two sets of readings and at the same time any induction effect in the leads will be automatically eliminated. Any component of the horizontal field threading the standard coil due to the same cause, will have a negligible effect, the main result being a small phase displacement of e_2.

(ii) *Error due to e.m.fs. set up in the potentiometer inductances.* A more important error is due to the component of the horizontal field threading the two self-inductances in the potentiometer. The effective areas of the inductances in the instrument constructed in the Royal College of Science were of the order of $2{\cdot}10^4$ cm.^{2}, which is about 2 per cent of the effective area of the search coils used in the field (about 10^6 cm.^{2}). If the two induced e.m.f.s are arranged to be equal and to act in opposition in the branch FG, figure 5, it can be shown that the currents in the various nets are given by a set of equations identical with equations (5) except

that e_1 is replaced by $(e_1 - e')$, where e' is the e.m.f. in the inductance. Consequently the instrument will give the components of this quantity, but by reversing the sign of e_1 the unwanted effect e' can be eliminated. The two e.m.f.s can be made equal by having the coils of identical construction and by placing them in parallel planes a short distance apart. Since they are to be joined in opposition their effective inductance is reduced to $(\frac{1}{2}L - M)$, where M is the mutual inductance between them. Consequently the condition that the currents in the two branches shall differ in phase by $\pi/2$ is now

$$L - 2M = CR^2.$$

By using astatically wound inductances the errors due to this source may be reduced to a negligible value and the instrument can be made to have a constant zero.

(iii) *Error due to the self-capacity of the search coils.* To a first approximation the self-capacity of the search coil of inductance L and resistance R may be represented by a condenser C' across its terminals, and it is the potential across this condenser which is measured. If the impedance of the condenser is large compared with the impedance of the coil the error will be small. For an error of less than 1 per cent $\frac{1}{\omega C'} > 100\{R^2 + L^2\omega^2\}^{\frac{1}{2}}$. With small coils this error will be negligible but may become important for the larger coils used to measure very small fields. Consequently the error due to the self-capacity of the coil will impose a limit to the lower range of the apparatus.

(iv) *Errors due to residuals in the potentiometer resistances.* If the residual inductances of the two potentiometers are small and equal it can be shown that to the first order of small quantities the two arms in parallel act as a non-inductive resistance R. Hence there is no change in the main current u in the circuit due to this cause. If μ is the reactance per ohm of the potentiometer, and v and w the currents in the two branches then

μ, v, w

$$u = e_2/(R + R_2)(1 + j),$$

$$uR = (R + Lj\omega + j\mu R).v = (R - j/\omega C + j\mu R).w$$

or $u = (1 + j + \mu j).v = (1 - j + \mu j).w$ at the normal frequency $\omega_0/2\pi$, and

$$e_1 = r(1 + j\mu).v - s(1 + j\mu).w;$$

whence, squares of terms involving powers of μ being neglected,

$$\frac{e_1}{e_2} = \frac{1}{2(R + R_2)}\left[\left\{s - \frac{\mu}{2}(r - s)\right\} + j\left\{r - \frac{\mu}{2}(r + s)\right\}\right] \quad \ldots\ldots(9).$$

The largest value that $(r - s)$ can attain is R, and hence if $\frac{1}{2}\mu R$ (half the residual of the potentiometer resistance) is made less than the smallest tapping on the potentiometer, the residual error will not appear in the measurement and the instrument will remain direct-reading.

(v) *Temperature corrections.* One branch of the potentiometer contains a large fraction of its resistance in the form of copper windings, and since a field instrument of this nature is exposed to a wide range of temperatures, errors will arise as a result

of the variation of resistance with temperature. It can be shown that a change in resistance of δ per cent will produce errors in the measurement of $\delta/4$ per cent, and except for extreme temperatures the errors will be negligible. They may be eliminated entirely with a variable compensating resistance calibrated in temperatures.

(vi) *Error due to change of frequency.* If the instrument has been made direct-reading for the normal working frequency $w_0/2\pi$, fluctuations in the frequency of the generator will alter the calibration and errors will be introduced. The general balance condition is given by equation (6) and hence for a frequency $(\omega_0 + \delta\omega)$

$$e_1/e_2 = \{r - j\omega_0 CR\,(1 + \delta\omega/\omega_0).s\}/[j\omega_0 CR\,(1 + \delta\omega/\omega_0)\{R_2 + L_2 j\omega_0\,(1 + \delta\omega/\omega_0)\} + R + R_2 + (L + L_2)\,j\omega_0\,(1 + \delta\omega/\omega_0)].$$

Since the instrument is calibrated for the frequency ω_0

$$R = L\omega_0 = 1/\omega_0 C;\ L_2\omega_0 = R + R_2,$$

if we neglect squares of small quantities the balance condition simplifies to

$$\frac{e_1}{e_2} = -\frac{1}{2\,(R - R_2)}\left[\left\{s + r\,\frac{\delta\omega}{\omega_0}\right\} + j\left\{r - \frac{\delta\omega}{\omega_0}\,(r + s)\right\}\right] \qquad \text{.....(10).}$$

If the two readings r and s are of the same order of 1 per cent change in the frequency will produce a 2 per cent error in the quadrature component and a 1 per cent error in the in-phase component. The lack of symmetry in the frequency correction is due to the fact that as well as a change in the magnitude of the currents in the two branches there is a phase-displacement of the main current relative to the induced e.m.f.

§ 7. DESCRIPTION OF THE APPARATUS

Two instruments on the principle have been constructed in the workshops of the Royal College of Science, and calibrated and tested in the Applied Geophysics Department. The first was constructed before it was realized that the e.m.f.s induced in the inductances by the alternating magnetic field to be examined would introduce serious error. In the tests the instrument was actually well removed from the magnetic field so that the error could be ignored, but in field practice this would not be possible. The current in these tests was obtained from a petrol-driven generator of the type used in the field, but it was found that the frequency fluctuated so rapidly and in such an erratic manner that the equation (10) could not be applied. As a result the measurements were only accurate to 3 or 4 per cent, owing chiefly to frequency-variations.

Accordingly a second instrument, embodying the design examined above for the compensation of the induced e.m.f.s in the inductances, was constructed. At the same time a synchronous motor was installed to drive the generator, and measurements showed that the frequency was constant to within 1 part in 500. The condensers used in the second instrument were good-quality waxed-paper

condensers* adjusted to values $2{\cdot}34_2\mu$F. and $2{\cdot}34_6\mu$F. The inductances were measured at a frequency of 530 c. sec. and had values $38{\cdot}8_3$mH. and $38{\cdot}8_1$mH. respectively, and their resistance in series with one another was $24{\cdot}38\Omega$. at 20° C. The total inductance in the circuit was adjusted to $76{\cdot}8_6$mH. by varying the mutual inductance between the parallel coils. The two 256·0-Ω. resistances were each composed of a potentiometer of 170Ω. together with fixed non-inductive resistances to give the correct total. The potentiometers were wound with 30 s.w.g. enamelled Eureka wire on thin strips of cross-section 1·5 cm. by 0·1 mm., each strip being wrapped round a circular former with a rotating arm. Each turn of wire had a resistance of about 0·2Ω. and subtended an angle of less than half a degree at the centre, so that readings could be made with this accuracy. The calculated inductance of the potentiometer was of the order of 0·05 mH. corresponding to a residual of 0·2Ω. at the working frequency.

A suitable pair of search coils for field work would consist of a horizontal coil of 200–300 turns on a frame about 60 cm. square, with a vertical coil of approximately the same dimensions. If the vertical coil is tapped so that differing numbers of turns can be employed, the range of the apparatus may be extended. The calibration of the system may be performed by setting R_2 to the value calculated from a knowledge of the frequency, or by a subsidiary horizontal coil in place of the vertical search coil. In the latter case the balance is secured by adjusting R_2 and the in-phase scale, the quadrature scale being set at zero.

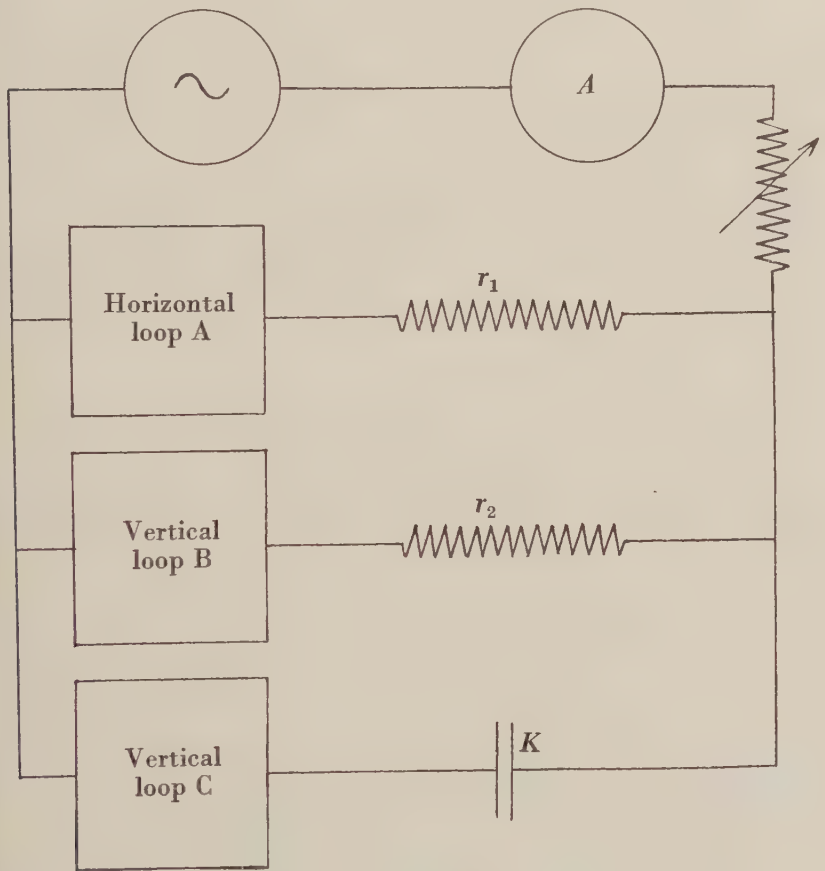
§ 8. TEST OF THE APPARATUS

As it was not possible to subject the apparatus to a rigorous field test, field conditions were simulated on a small scale in the laboratory. An elliptically polarized field, whose components could be varied in amplitude and phase, was produced by means of three large loops. Each loop was 7 ft. square, bisected the other two at right angles, and carried a current generally differing in amplitude and phase from the currents in the other two. Consequently the nature of the elliptically polarized field at the centre could be controlled completely. Search coils of much smaller size than those employed in the field were placed at the centre so that the magnetic field at all points in the small volume used in the experiment could be assumed to be spatially uniform. The search coils had diameters of 8 in. and the calculated magnetic field was uniform to within 2 per cent. The horizontal search coil with 530 turns of wire had an inductance of 92·7 mH. and effective area of $1{\cdot}97 \times 10^5$ cm², while the vertical coil of 15 mH. had an effective area of $7{\cdot}5 \times 10^4$ cm². Their self-capacities were less than $0{\cdot}003\mu$F., and hence the errors due to this cause may be neglected. The balance readings were obtained by using a resistance-capacity-coupled valve amplifier as the detector, with a filter circuit in the output to reduce the effect of harmonics.

* In the original instrument these were mica condensers supplied by the Radio Research Board, Slough, through the courtesy of the Department of Scientific and Industrial Research. They were replaced by waxed-paper condensers to reduce the weight of a field instrument.

The circuit used to produce the magnetic field is shown in figure 6. The loops were made with aluminium-armoured wire with the covering earthed. The loops and the search coils virtually constitute a screened transformer in which capacity currents are reduced to a minimum, but the effective current will differ from the core current on account of the eddy currents set up in the shield. Since this is the same for all loops, then if I_1, I_2 and I_3 are the currents in the three loops I_1, I_2, I_3

$$I_1 r_1 = I_2 r_2 = -jI_3/\omega K$$



r_1

r_2

K

Figure 6.

or since the magnetic fields are proportional to the currents,

$$Zr = Xr_2 = -jY/\omega K \qquad \text{.....(11)},$$

where Z is the vertical and X and Y the two horizontal components of the field. Hence X and Z are in phase and Y and Z are in quadrature, while X and Y, besides being in quadrature, differ in direction by 90°, since the coils producing them are at right angles. The test consisted of measuring X and Y in terms of Z to see if the instrumental readings satisfied the above conditions. X, Y, Z

It was found impossible to make the test directly on account of the residual field due to the leads to the loops and the phase-splitting device; moreover, under these conditions the apparatus could not be calibrated experimentally. Hence R_2 was set to the calculated value for the frequency used. With the horizontal loop permanently in the circuit, the horizontal field was measured for various azimuths of the vertical search coil. The tests consisted of the following operations. (*a*) Measurement of residual field R. (*b*) Measurement of the combined field of R and field X due to B. (*c*) Measurement of the combined field of R and field Y due to C. (*d*) Measurement of the combined field of R and the fields due to B and C. The balances obtained were sharp and could be recovered to within half a scale-division.

In all cases it was found that the horizontal field could be represented by an equation of the type

$$\alpha = a + b \cos(\theta + \delta)$$

θ, α where θ is the azimuth of the vertical coil and α the measure of the component in terms of the scale-readings on the potentiometer. The first term is the error (i) discussed on p. 357, while the second is the contribution of the horizontal field.

$\alpha_1, \alpha_2, \alpha_3$ From readings $\alpha_1, \alpha_2, \alpha_3$ obtained in azimuths θ, $(\theta + 120)$ and $(\theta - 240)$ the three constants may be calculated.

$$a = \tfrac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)$$

$$b = \tfrac{1}{3}\{3(\alpha_2 - \alpha_3)^2 + (\alpha_2 + \alpha_3 - 2\alpha_1)^2\}^{\frac{1}{2}}$$

$$\delta = \tan^{-1}\{(\alpha_2 - \alpha_3)\sqrt{3}/(\alpha_2 + \alpha_3 - 2\alpha_1)\} - \theta.$$

A typical set of readings illustrating the accuracy which can be obtained will now be given.

Measurement of total field (vector sum of R, X and Y).

$r_1 = 5{\cdot}55\,\Omega$, $r_2 = 10{\cdot}09\,\Omega$, $K = 10{\cdot}2\,\mu$F., $1/\omega K = 29{\cdot}3\,\Omega$.
Total current, 2·56 A. Frequency, 533 c./sec.

Azimuth θ	Readings at θ	Readings at $(\theta + 120)$	Readings at $(\theta + 240)$	a	b	$(\theta + \delta)$ (degrees)	δ (degrees)
In-phase component							
0°	154	− 98	− 54	0·7	154·6	9·9	9·9
30°	120·5	− 145	26	0·5	154·9	39·4	9·4
60°	55	− 151	99·5	1·2	154·5	69·6	9·6
90°	− 23·5	− 117·5	146·5	1·8	154·5	99·4	9·4
			Mean	1·0	154·5	—	9·6
Quadrature component							
0°	− 54·5	− 61·5	116·5	− 0·2	116·2	− 62·0	− 62·0
30°	− 97·5	− 3·5	102	− 0·3	115·2	− 31·9	− 61·9
60°	− 116·5	55	61	− 0·2	116·2	− 1·9	− 61·9
90°	− 102·5	99	3·5	—	116·3	28·3	− 61·7
			Mean	—	116·0	—	− 61·9

One division corresponds to a ratio of $|e_1 e_2|$ equal to $8{\cdot}80 \cdot 10^{-4}$ on the in-phase scale and to $8{\cdot}60 \cdot 10^{-4}$ on the quadrature scale. The in-phase and quadrature

components of the fields being denoted by the suffixes i and q, the equations representing the horizontal fields for the above current-distribution were *i, q*

$$\left.\begin{aligned} R_i &= 1 - 91\cos(\theta - 14{\cdot}4^\circ) \\ R_q &= -\,55{\cdot}5\cos(\theta - 19{\cdot}1^\circ) \end{aligned}\right\} \quad \ldots\ldots(a),$$

$$\left.\begin{aligned} R_i + X_i &= 1 + 155{\cdot}5\cos(\theta + 9{\cdot}5^\circ) \\ R_q + X_q &= -\,55{\cdot}5\cos(\theta - 18{\cdot}8^\circ) \end{aligned}\right\} \quad \ldots\ldots(b),$$

$$\left.\begin{aligned} R_i + Y_i &= 1 - 90{\cdot}5\cos(\theta - 14^\circ) \\ R_q + Y_q &= -\,117\cos(\theta - 62{\cdot}3^\circ) \end{aligned}\right\} \quad \ldots\ldots(c),$$

$$\left.\begin{aligned} R_i + X_i + Y_i &= 1 + 154{\cdot}5\cos(\theta + 9{\cdot}6^\circ) \\ R_q + X_q + Y_q &= -\,116\cos(\theta - 61{\cdot}9^\circ) \end{aligned}\right\} \quad \ldots\ldots(d).$$

From equations (a), (b) and (c) we derive

$$\left.\begin{aligned} R_i + X_i + Y_i &= 1 + 155{\cdot}5\cos(\theta + 9{\cdot}1^\circ) \\ R_q + X_q + Y_q &= -\,117\cos(\theta - 62{\cdot}2^\circ) \end{aligned}\right\} \quad \ldots\ldots(e),$$

which are in good agreement with the measured values (d). The two fields due to the vertical loops B and C are, from the above equations, given by

$$\left.\begin{aligned} X_i &= 241{\cdot}5\cos(\theta + 0{\cdot}6^\circ) \\ X_q &= 0{\cdot}5\cos(\theta + \gamma_1) \end{aligned}\right\}$$

$$\left.\begin{aligned} Y_i &= 0{\cdot}5\cos(\theta + \gamma_2) \\ Y_q &= 85{\cdot}5\cos(\theta + 91{\cdot}3^\circ) \end{aligned}\right\}.$$

Of these, the quadrature component X_q* is small compared with X_i while the in-phase component Y_i is small compared with Y_q. Hence the field due to B is in phase with that due to A and in quadrature with that due to C. Further, the angle between the two fields is 90·7°.

The ratios of the amplitudes of the various fields may be calculated from the known current-distribution by means of equation (11), and also from the scale-values of the instrument-readings. The results for the experiments quoted are as follows.

Ratio	Z/X	Y/X
Calculated from current-distribution	1·82	0·345
Calculated from instrument readings	1·81	0·346

Hence the fields as measured by the instrument agree well with the fields calculated from the known current-distribution in the loops quantitatively as well as qualitatively.

The effect of an external field on the instrument itself was examined also to determine if the compensation of the e.m.f.s set up in the potentiometer inductances was sufficient. The field in which the instrument was placed was produced by two subsidiary square coils of side 92 cm. separated by a distance of 50·2 cm. This separation is the most favourable for producing a minimum variation of the field

* The terms X_q, Y_i are small differences between large quantities and it is doubtful if the figures have any real significance. For this reason γ_1 and γ_2 are not given.

parallel to the axis over a plane midway between the coils*. With the search coils in a fixed position in the system for producing an elliptically polarized field, the measuring instrument was moved into the magnetic field produced by the subsidiary coil system with the two inductances symmetrically on either side of the mid-plane. The readings representing the ratio of the fields threading the search coils were found to agree among themselves within 0·5 per cent, although the induced e.m.f.s in the inductances produced zero-shifts of from 0 to 50 divisions on one scale, and from 0 to 20 divisions on the other. The displacement of the zero, which is a measure of the magnetic fields threading the inductances, when plotted against the position of the instrument in the magnetic field followed closely the transverse variation of a magnetic field due to the square-coil system. Further, it was found that if the inductances were placed in unequal fields the instrument readings no longer agreed with the values obtained when the instrument was in zero field. Consequently the compensation is not perfect unless the inductances are in equal fields.

§ 9. ACKNOWLEDGMENTS

In conclusion, I wish to thank Prof. A. O. Rankine for the interest he has shown in this work; and the Department of Scientific and Industrial Research for the financial assistance which rendered it possible.

* A. O. Rankine, "Presidential Address to the Physical Society," *Proc. Phys. Soc.* **46**, 1 (1934).

517.391

NOTE ON INTEGRALS OF PRODUCTS OF EXPERIMENTALLY DETERMINED MAGNITUDES

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ABSTRACT. The integral of a product of quantities known only for discrete values of a variable is given correctly by the simple sum of the products for uniformly distributed values of the variable. Nothing is gained by increasing the number of component products beyond the number of observed values of either factor.

A problem which may arise in various branches of physics may be illustrated by an optical example. It is desired to find the luminosity of the light from a given source transmitted by a filter as evaluated by a "standard observer." The quantity required is

$$\int etv\,d\lambda$$

taken over the whole visible spectrum, where e represents the energy of the source, t is the transmission factor of the filter, and v is the visibility function by which the standard observer is defined, all expressed as functions of the wave-length λ. Of these magnitudes only e is in any case known precisely for all wave-lengths. The value of t will be determined to a certain accuracy for discrete wave-lengths spread over the spectrum, and v is also defined by a table of numbers for discrete wave-lengths only. For the integral to have a meaning it is necessary to suppose that when any factor is not a known continuous function of the wave-length we are to understand by the symbol a continuous function of simple type which assumes the known values for definite wave-lengths. In effect this means that these factors will be regarded as polynomials in which the coefficients may change with the wave-length. The changes in the coefficients are due to the adoption of polynomials of much lower order than the number of wave-lengths for which the functions are defined. In consequence of this change, and of the substitution of summation for integration, the value found for the integral will vary with the detailed procedure adopted. For example the result obtained by interpolating values of the product according to a given rule will differ from that consequent on applying the same interpolation rule to the individual factors of the product. If the wave-length subdivision is sufficiently close the range over which the integral can vary will be without real significance. On the other hand the labour of experimental observation and subsequently that of computation are directly proportional to the number of terms involved in the sum. It is thus important to know on what these variations depend, and to determine what procedure yields an answer, free from errors exceeding a selected value, with the least expenditure of labour.

In optical problems it most frequently happens that the integrand consists effectively of two factors. In the example given e is much more frequently than not the energy emitted by a standard source, and the product ev is then tabulated. It is quite sufficient for the discussion of the problem to consider only two factors. The extension to a greater number of factors can readily be made if desired by treatment similar to that given here.

We suppose then that the integrand a is the product of two factors b and c which are only known for positive and negative integral values of the variable x over the range for which integration is to be made. For all values of x outside this range we ascribe the value zero to one at least of the factors b, c. In the neighbourhood of any particular integral value of x, which we may without loss of generality suppose to be $x = 0$, we may represent b and c by polynomials in x. If we represent by α, β, γ the central difference operators defined by the equations

$$4\alpha a_p = a_{p+1} - 2a_p + a_{p-1} = b_{p+1}c_{p+1} - 2b_pc_p + b_{p-1}c_{p-1}$$
$$4\beta b_p = b_{p+1} - 2b_p + b_{p-1}$$
$$4\gamma c_p = c_{p+1} - 2c_p + c_{p-1},$$

where p is zero or a positive or negative integer, the polynomial of order $2n$ representing b in the neighbourhood of $x = 0$ determined from its $2n + 1$ values $b_n, b_{n-1}, \ldots b_{-n}$ is

$$b = \left\{1 + \frac{x^2}{2!}\cdot 4\beta + \frac{x^2(x^2-1^2)}{4!}(4\beta)^2 + \frac{x^2(x^2-1^2)(x^2-2^2)}{6!}(4\beta)^3 + \ldots\right\} b_0$$
$$+ \tfrac{1}{2}\left\{x + \frac{x(x^2-1^2)}{3!}4\beta + \frac{x(x^2-1^2)(x^2-2^2)}{5!}(4\beta)^2 + \ldots\right\}(b_1 - b_{-1}),$$

where the series in the first bracket terminates after $n + 1$ terms and that in the second after n terms, and a similar expression holds for c. This is merely one form of the familiar Lagrangian interpolation formula, as may be verified on inspection. We can thus express the integrand as a polynomial in x of order $4n$, where the coefficients are the values of b and c for integral values of x, and we are assuming that each factor is to be interpolated separately. Now in every interval the integral is of the form

$$\int_{-\xi}^{\xi} bc\,dx,$$

where ξ is the same for all the intervals. Clearly the contribution of odd powers of x in the product bc is zero. We may accordingly reject in bc all terms with odd powers of x, and take as the integrand

$$\left\{1 + \frac{x^2}{2!}\cdot 4\beta + \frac{x^2(x^2-1^2)}{4!}(4\beta)^2 + \ldots\right\}\left\{1 + \frac{x^2}{2!}4\gamma + \frac{x^2(x^2-1^2)}{4!}(4\gamma)^2 + \ldots\right\} b_0c_0$$
$$+ \frac{x^2}{4}\left\{1 + \frac{x^2-1^2}{3!}4\beta + \ldots\right\}\left\{1 + \frac{x^2-1^2}{3!}4\gamma + \ldots\right\}(b_1 - b_{-1})(c_1 - c_{-1}).$$

Also

$$(b_1 - b_{-1})(c_1 - c_{-1}) = 2(b_1c_1 - 2b_0c_0 + b_{-1}c_{-1}) - 2(b_1 - 2b_0 + b_{-1})c_0$$
$$- 2b_0(c_1 - 2c_0 + c_{-1}) - (b_1 - 2b_0 + b_{-1})(c_1 - 2c_0 + c_{-1})$$
$$= 8(\alpha - \beta - \gamma - 2\beta\gamma)\,b_0c_0,$$

so that all the coefficients in the integral, apart from the term b_0c_0 itself, involve the central differences of b_0, c_0 or b_0c_0. The integral may therefore be evaluated by applying certain factors to the sums of terms involving these central differences. If however the central differences are themselves of suitably small magnitude all sums in which they appear will be negligible, and the integral itself will then be accurately determined by the simple sum $\Sigma b_p c_p$. The lowest order of the central differences which appear as a correction to this sum is clearly a matter of interest.

Since β and γ act on different factors it is obvious that these operations are commutable for multiplication. If we extend the definition of α and define a new operator δ in accordance with the equations

$$4\alpha b_p c_q = b_{p+1}c_{q+1} - 2b_p c_q + b_{p-1}c_{q-1}$$
$$4\delta b_p c_q = b_{p+1}c_{q-1} - 2b_p c_q + b_{p-1}c_{q+1},$$

it is clear, since the operators only involve differencing and additions to suffixes, that we may commute the order of any of the operators α, β, γ, δ in products. It is also easy to verify that

$$(\beta - \gamma)^2 b_p c_q = \alpha\delta b_p c_q.$$

Moreover, from the definition of α it follows that on summation any expression including the operator α as a factor vanishes if the operand vanishes at the limits of summation. In evaluating the integral we may therefore reject all terms involving α, and by the relation just given also all terms containing $(\beta - \gamma)^2$ as an operator.

Now, apart from b_0c_0, the terms of lowest order in the expansion of a in powers of β and γ are

$$\left\{\frac{x^2}{2!}4(\beta+\gamma) - \frac{x^2}{4}8(\beta+\gamma)\right\} b_0c_0,$$

and vanish identically. Those of the next order are found to be

$$\frac{4x^2(x^2-1^2)}{3!}\{2\alpha(\beta+\gamma) - (\beta-\gamma)^2\} b_0c_0,$$

and therefore vanish on summation. This result suggests that when differences of all orders are taken into account all the terms except the first may disappear on summation. To investigate this we note that if we write o, $\sinh^2\theta$ and $\sinh^2\phi$ for α, β and γ respectively the effective operator acting on b_0c_0 is represented by

$$\cosh 2x\theta \cosh 2x\phi - 2\sinh 2x\theta \sinh 2x\phi\,(\sinh^2\theta + \sinh^2\phi + 2\sinh^2\theta \sinh^2\phi)/\sinh 2\theta \sinh 2\phi$$

or by

$$\frac{\cosh 2x(\theta-\phi)\sinh^2(\theta+\phi) - \cosh 2x(\theta+\phi)\sinh^2(\theta-\phi)}{\sinh 2\theta \sinh 2\phi}$$

from which it is clear that every term of the expansion involving x contains $(\beta - \gamma)^2$ as a factor, and thus vanishes on summation. It follows that any departure of the integral from the value $\Sigma b_p c_p$ is attributable to the termination of the series occurring in the integrand, i.e. to approximations in the interpolation differing from the best values obtainable. The correct value of the integral must therefore be taken to be $\Sigma b_p c_p$, and any attempts to improve accuracy by finding intermediate values of the integrand introduce errors when the change is not precisely nil.

It will be noted that no attention has been paid to the value to be given to ξ. The result just reached shows that this is unnecessary, and also removes all formal difficulties which require consideration when brief interpolation formulae are applied, such as discontinuities in the function, or in its differential coefficients, when x differs from an integer by ξ.

The fundamental assumption upon which this simple conclusion rests is that interpolation is effected by a symmetrical Lagrangian formula. We have taken as an illustration the case when the number of given points is odd, but the modifications needed when the number is even are slight and readily made. It follows that interpolation of a different type in problems of this kind should not be substituted for the Lagrangian unless this is justified by special considerations.

In practice it will often happen that observations are not available for uniformly distributed values of one of the variables. For instance the transmission factor of a filter might conceivably be obtained for certain wave-lengths of a line spectrum. The correct procedure is then to construct a Lagrangian expression for the given observations and evaluate the transmission from this for a series of uniformly distributed points.

There is only one case in which there is any need to subdivide any of the standard intervals. This is when the values of the observed magnitude (represented by c say) are not adequately defined by their values at the standard points. An extreme example is afforded by a function which is zero at these points and is positive elsewhere. The best procedure is then to subdivide the standard interval only where and only to the extent it is necessary to secure a proper representation of the function measured, to interpolate by the Lagrangian method for the sub-intervals, and to separate the integral into two parts as considered in the following argument.

Let the whole range of integration be represented by $ABCD$, BC being the region where it is essential to subdivide the standard intervals. We can divide c into two parts c_1 and c_2, c_1 being equal to c outside the range BC, and being defined within the range BC precisely by its values at the standard intervals. Under these conditions if we write $a_1 = bc_1$, $a_2 = bc_2$, we obtain exactly the same value for the a_1 sum whether we take the larger or the smaller intervals, and this value is equal to $\int_A^D a_1 dx$. Again a_2 is a function by hypothesis accurately defined by its values at the smaller intervals within the range BC, and vanishing outside this range. Hence

$$\int_A^D a_2 dx = \int_B^C a_2 dx$$

and both are equal to the a_2 sum for the smaller intervals. Also

$$\int_A^D a dx = \int_A^D a_1 dx + \int_A^D a_2 dx$$

and the total sum therefore represents the integral correctly.

To determine the a_2 sum we clearly have to interpolate b for the intermediate points in BC, and also to interpolate c_1 within this interval in order to obtain c_2 by

subtraction from c. In practice this work will be light, for in subintervals the changes in b and c_1 will tend to be simple in character. In obtaining the a_2 sum care must be taken not to omit division by the number representing the ratio in which the intervals have been increased, which takes proper account of the reduced value of the new intervals.

From this discussion it immediately follows that there is no point in evaluating an integral by summing more terms than there are observed values of either factor in the integrand. For example, if observations are made on c corresponding only to alternate entries of the standard function b, it will be better to construct an equivalent b table for the larger intervals than to interpolate c for the standard intervals. By the theory just considered it is apparent that the new b table can be constructed to give exactly the same result as precise interpolation of c would yield, but the labour involved is reduced to less than a half. The work involved in constructing the new table is comparable with that required for a single interpolation of the factor c for the complete range. It is clearly advantageous to construct tables for all the observation intervals in regular use.

DISCUSSION

Mr J. Guild. This paper deals with a type of problem which arises in most branches of experimental physics. Integration of a product of two or more quantities each of which is a function of some variable (wave-length in the example chosen) is of frequent occurrence, and the experimental information concerning the various functions is very often available only for a discrete series of values of the variable. There are interesting questions of principle involved in the status of interpolated values of a quantity when (*a*) the quantity in question has no existence except as *defined* by its values at specified values of the variable, as in the case of the various quantities, such as v_λ in the author's example, defining the hypothetical "standard observer"; and (*b*) when the quantity is a real physical quantity, such as energy, or the transmission of a filter (e_λ and t_λ in the author's example) which is defined independently of the variable with respect to which interpolation is carried out, and which possesses an actual value at any value of the variable whether we know what it is or not. These questions would carry us outside the reasonable scope of this discussion: I only mention that they exist, because there is a tendency among some physicists to assume that mathematical considerations can lead to principles of interpolation which will obtain, from a discrete series of observations of a physical quantity, the physically correct values at intermediate points, and it is necessary always to bear in mind that any process of interpolation, applied to physical quantities, necessarily yields only an approximation—it is a make-shift device for getting certain information from inadequate data. Interpolation can never be regarded as a formal substitute for a complete statement of the physical facts, where such physical facts exist as in the cases coming under (*b*); whereas it can be regarded as a formal substitute for a more extensive definition of quantities coming under (*a*), for which no independent physical facts exist.

Mr Smith does not draw this distinction in the paper, but treats the interpolates of e and t (or their product) as having the same significance as those of v. In practice, of course, this must be done, since the problem is, by hypothesis, how best to proceed when one is ignorant of the intermediate values of the physical quantities, and our actual procedure cannot be affected by the existence or non-existence of true values which are, in any case, outside our knowledge. The importance of the distinction is simply in the significance we attach to the result to which the procedure leads.

Mr Smith's procedure is, in effect, to extend the *definition* of the standard observer so that it includes a Lagrangian interpolation formula as defining values of v_λ between the tabulated values, and to *assume* that the variation of the physical quantity $e_\lambda t_\lambda$ between the observed values will also be in accordance with this formula. From these premises he obtains two results of great practical importance, which, as far as I am aware, have not been stated before and which are certainly not generally known, namely, that the correct value of the integral is simply the sum of the products of the various factors at the discrete (equally spaced) values of the variable for which their values are known, and that there is no point in evaluating an integral by summing more terms than there are observed values of any of the factors in the integrand.

Applied to practical problems the first conclusion must be regarded as approximate, inasmuch as we have no guarantee that the actual intermediate values of the physical factors are correctly represented by Lagrangian interpolation. This of course is shown by the examples illustrating the author's next paper, in which the results obtained by applying these formulae over two different ranges of interpolation, though agreeing very closely, do not agree exactly.

It must, nevertheless, be regarded as the best value obtainable from a given set of data, because although we know that the Lagrangian formula is almost certainly inaccurate in any particular case, we have absolutely no reason for expecting the unknown facts to be better represented by any other formula. There is therefore nothing we can do with the data which will reduce the uncertainty of the result, and Mr Smith's second conclusion "that there is no point etc. . . ." is universally true whatever may be the actual (but by hypothesis unknown) values of the quantities between the observed points. I have before now gone to great trouble to interpolate between observed spectrophotometric values in order to get what I hoped would be a more reliable value for the integral photometric transmission, or the colour coordinates of a filter, and doubtless others have done the same. Clearly this is pure waste of labour and adds nothing to the reliability of the result, which can only be improved, when improvement is necessary, by diminishing the interval at which experimental observations are made.

Author's reply. I agree with Mr Guild on the importance of realizing the precise significance of the procedure followed in calculations, but I am not sure that our views on some of the points he refers to are in complete agreement though I think we should agree up to the point where the question becomes only of theo-

retical interest. There is no doubt that any finite continuous function representing physical magnitudes or the mathematical functions needed by the physicist can be represented over any conveniently short range of a variable by a polynomial with an error not exceeding an arbitrarily assigned small but finite value, and unless the physical magnitudes are of this character an integral involving them has no meaning. As the observational intervals are reduced the possible range of variation of the function within the interval is reduced continuously also, and the two tend to zero together. In any case in which this did not occur I should incline to the view that the variations represented experimental errors, and take smoothed values rather than the irregular observations as the proper representation of the magnitude. That is to say there is a point beyond which I should not admit that data were inadequate or the smooth function an approximation. I can see no possible ultimate escape from reliance upon interpolation for physical magnitudes in problems of this kind. From their very nature the complete statement of the physical facts in the sense I think Mr Guild means is always impossible, so that some such view as the one I have given is necessary if we are to deal with physical measurements in this connexion at all. If then we leave on one side, as we must, all questions of the adequacy of particular sets of data, we have only to consider the relative advantages and conveniences of polynomials constructed in various ways (all of which may represent the data within given limits), e.g. whether we should use an expression of a particular order, and whether we ought to secure osculatory contact where the sections meet. In view of the result reached in this paper with the formula first considered there is no point in discussing any other.

It ought to be particularly remembered that the conclusions reached in this paper depend upon the assumption that the integrand vanishes for an indefinitely long range of the variable at either end of the region for which it differs from zero. It is for this reason that we obtain, on replacing one of the factors by unity, a formula differing from the well-known expressions for evaluating areas exemplified by Simpson's and Weddle's rules; in these it is assumed that nothing is known of the function outside the range of integration.

(*Discussion continued on page* 478)

535.62

CONDENSED TABLES FOR COLOUR COMPUTATION

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ABSTRACT. It is sometimes sufficient in the spectrophotometry of coloured materials for the determination of their colour coordinates on the C.I.E. system to take measurements at intervals of 10 mμ. instead of the standard interval of 5 mμ. Special tables have been computed for use in these cases which give with less labour results indistinguishable from those which would be obtained from the use of the standard observer tables.

The C.I.E. system for the specification of colours has been discussed at some length in the *Transactions of the Optical Society**. In addition to the specification of scales on which colour coordinates are to be expressed that system specifies some standard illuminants for colorimetry and also a standard observer. The most precise specification of a colour is obtained by accurate and adequate spectrophotometry followed by computation utilizing the tabulated properties of the standard observer. By adequate spectrophotometry is to be understood measurement at wave-length intervals small enough to represent the transmissive or other properties of the material examined with accuracy over the whole visible spectrum. Since the standard tables proceed by intervals of 5 mμ. it will be understood that for the general run of sources and materials measurements at these intervals will suffice. This of course does not imply that there are not unusual cases in which measurement at still smaller intervals is needed: when this occurs intermediate values in the standard observer tables will have to be interpolated; it is not then necessary to specify the interpolation formula to be used since the tabulated values are close enough for significant variations not to arise.

Of greater importance than the reduction of intervals is the problem of extending the intervals and so reducing the amount of observational work as well as of computation needed to obtain a correct result. Experience shows that the spectral properties of many materials are adequately described by their values at intervals of 10 mμ. The formally correct procedure when this is known to be true is then to measure at the 10 mμ. intervals, to interpolate values for the 5 mμ. intervals by a suitable formula, and then to compute exactly as though observations had been made at the 5 mμ. intervals. It is possible however to obtain the same result with less labour by constructing special tables to replace the standard tables *under these special circumstances only*. It has been shown elsewhere† in this volume that we

* The C.I.E. Colorimetric Standards and their Use. *Trans. Opt. Soc.* **33**, 73 (1931–2).
† Page 365.

Table 1

Equal energy stimulus			Wave-length (mμ.)	Standard illuminant A		
$\bar{x}$	$\bar{y}$	$\bar{z}$		$E\bar{x}$	$E\bar{y}$	$E\bar{z}$
0·0023	0·0000	0·0106	**380**	0·0010	0·0000	0·0048
0·0082	0·0002	0·0391	**390**	0·0046	0·0001	0·0219
0·0283	0·0007	0·1343	**400**	0·0193	0·0005	0·0916
0·0840	0·0023	0·4005	**410**	0·0688	0·0019	0·3281
0·2740	0·0082	1·3164	**420**	0·2666	0·0080	1·2811
0·5667	0·0232	2·7663	**430**	0·6479	0·0265	3·1626
0·6965	0·0458	3·4939	**440**	0·9263	0·0609	4·6469
0·6730	0·0761	3·5470	**450**	1·0320	0·1167	5·4391
0·5824	0·1197	3·3426	**460**	1·0207	0·2098	5·8584
0·3935	0·1824	2·5895	**470**	0·7817	0·3624	5·1445
0·1897	0·2772	1·6193	**480**	0·4242	0·6198	3·6207
0·0642	0·4162	0·9313	**490**	0·1604	1·0398	2·3266
0·0097	0·6473	0·5455	**500**	0·0269	1·7956	1·5132
0·0187	1·0077	0·3160	**510**	0·0572	3·0849	0·9674
0·1264	1·4172	0·1569	**520**	0·4247	4·7614	0·5271
0·3304	1·7243	0·0841	**530**	1·2116	6·3230	0·3084
0·5810	1·9077	0·0408	**540**	2·3142	7·5985	0·1625
0·8670	1·9906	0·0174	**550**	3·7329	8·5707	0·0749
1·1887	1·9896	0·0077	**560**	5·5086	9·2201	0·0357
1·5243	1·9041	0·0042	**570**	7·5710	9·4574	0·0209
1·8320	1·7396	0·0032	**580**	9·7157	9·2257	0·0170
2·0535	1·5144	0·0023	**590**	11·5841	8·5430	0·0130
2·1255	1·2619	0·0016	**600**	12·7103	7·5460	0·0096
2·0064	1·0066	0·0007	**610**	12·6768	6·3599	0·0044
1·7065	0·7610	0·0003	**620**	11·3577	5·0649	0·0020
1·2876	0·5311	0·0000	**630**	8·9999	3·7122	0·0000
0·8945	0·3495	0·0000	**640**	6·5487	2·5587	0·0000
0·5681	0·2143	0·0000	**650**	4·3447	1·6389	0·0000
0·3292	0·1218	0·0000	**660**	2·6234	0·9706	0·0000
0·1755	0·0643	0·0000	**670**	1·4539	0·5327	0·0000
0·0927	0·0337	0·0000	**680**	0·7966	0·2896	0·0000
0·0457	0·0165	0·0000	**690**	0·4065	0·1467	0·0000
0·0225	0·0081	0·0000	**700**	0·2067	0·0744	0·0000
0·0117	0·0042	0·0000	**710**	0·1108	0·0398	0·0000
0·0057	0·0020	0·0000	**720**	0·0556	0·0195	0·0000
0·0028	0·0010	0·0000	**730**	0·0280	0·0100	0·0000
0·0014	0·0006	0·0000	**740**	0·0144	0·0062	0·0000
0·0006	0·0002	0·0000	**750**	0·0063	0·0021	0·0000
0·0003	0·0001	0·0000	**760**	0·0032	0·0011	0·0000
0·0001	0·0000	0·0000	**770**	0·0011	0·0000	0·0000
21·3713	21·3714	21·3715	Totals	109·8450	100·0000	35·5824

Table 1 (*continued*)

Standard illuminant B			Wave-length (mμ.)	Standard illuminant C		
$E\bar{x}$	$E\bar{y}$	$E\bar{z}$		$E\bar{x}$	$E\bar{y}$	$E\bar{z}$
0·0025	0·0000	0·0113	380	0·0036	0·0000	0·0164
0·0123	0·0003	0·0585	390	0·0183	0·0004	0·0870
0·0558	0·0014	0·2650	400	0·0841	0·0021	0·3992
0·2091	0·0057	0·9970	410	0·3180	0·0087	1·5159
0·8274	0·0248	3·9750	420	1·2623	0·0378	6·0646
1·9793	0·0810	9·6617	430	2·9913	0·1225	14·6019
2·6889	0·1768	13·4883	440	3·9741	0·2613	19·9357
2·7460	0·3105	14·4729	450	3·9191	0·4432	20·6551
2·4571	0·5050	14·1020	460	3·3668	0·6920	19·3235
1·7297	0·8018	11·3825	470	2·2878	1·0605	15·0550
0·8629	1·2609	7·3655	480	1·1038	1·6129	9·4220
0·2960	1·9190	4·2939	490	0·3639	2·3591	5·2789
0·0437	2·9133	2·4552	500	0·0511	3·4077	2·8717
0·0810	4·3669	1·3694	510	0·0898	4·8412	1·5181
0·5405	6·0602	0·6709	520	0·5752	6·4491	0·7140
1·4555	7·5959	0·3705	530	1·5206	7·9357	0·3871
2·6899	8·8322	0·1889	540	2·7858	9·1470	0·1956
4·1838	9·6060	0·0840	550	4·2833	9·8343	0·0860
5·8385	9·7722	0·0378	560	5·8782	9·8387	0·0381
7·4723	9·3341	0·0206	570	7·3230	9·1476	0·0202
8·8406	8·3947	0·0154	580	8·4141	7·9897	0·0147
9·7329	7·1777	0·0109	590	8·9878	6·6283	0·0101
9·9523	5·9086	0·0075	600	8·9536	5·3157	0·0067
9·4425	4·7373	0·0033	610	8·3294	4·1788	0·0029
8·1290	3·6251	0·0014	620	7·0604	3·1485	0·0012
6·2135	2·5629	0·0000	630	5·3212	2·1948	0·0000
4·3678	1·7066	0·0000	640	3·6882	1·4411	0·0000
2·8202	1·0638	0·0000	650	2·3531	0·8876	0·0000
1·6515	0·6110	0·0000	660	1·3589	0·5028	0·0000
0·8796	0·3223	0·0000	670	0·7113	0·2606	0·0000
0·4602	0·1673	0·0000	680	0·3657	0·1329	0·0000
0·2218	0·0801	0·0000	690	0·1721	0·0621	0·0000
0·1065	0·0384	0·0000	700	0·0806	0·0290	0·0000
0·0538	0·0193	0·0000	710	0·0398	0·0143	0·0000
0·0253	0·0089	0·0000	720	0·0183	0·0064	0·0000
0·0120	0·0043	0·0000	730	0·0085	0·0030	0·0000
0·0058	0·0025	0·0000	740	0·0040	0·0017	0·0000
0·0024	0·0008	0·0000	750	0·0017	0·0006	0·0000
0·0012	0·0004	0·0000	760	0·0008	0·0003	0·0000
0·0004	0·0000	0·0000	770	0·0003	0·0000	0·0000
99·0915	100·0000	85·3094	Totals	98·0699	100·0000	118·2216

cannot improve the accuracy of the integral of a product by interpolation between observed values of a factor. If both factors of the product are smooth to a very high order we obtain the best result by summing the products for the observed intervals. It follows that if the properties of the standard observer were smooth to a very high order we could not do better than simply employ the tabulated values corresponding to the wave-lengths for which observations have been made and form our sums without more ado. In fact the standard observer properties are smooth enough for the outstanding irregularities to produce no difference that could be appreciated in any tests that have so far been made at the National Physical Laboratory, but they do not formally satisfy the conditions to the highest order that might be considered. It has therefore been thought desirable to construct special tables which meet this condition fully.

That the problem can be solved by the construction of special tables is easily seen. Let b represent an entry in the tables specifying the standard observer, and c the corresponding experimentally determined quantity, for instance the transmission factor of a filter for the wave-length to which b refers. The quantity to be found is Σbc extending over all the wave-lengths for which b is tabulated. For some of these wave-lengths c is observed: let us distinguish these by adding a suffix o. For others values of c are obtained by interpolation among the observed c_0's: distinguish these by the suffix e. Then

$$\Sigma bc = \Sigma b_0 c_0 + \Sigma b_e c_e.$$

But c_e is given a value that may be denoted by $\Sigma w_0 c_0$, where w_0 is the weight given in the interpolation formula to c_0 in the evaluation of c_e. Thus on rearranging we have

$$\Sigma bc = \Sigma b_0' c_0,$$

where

$$b_0' = b_0 + \Sigma b_e w_0.$$

The quantity b_0' is a typical entry of the new condensed table, and it is clear that the contribution to b_0' from the excluded entries b_e is the value of b derived by high order interpolation from the set b_e alone.

In constructing the tables which follow the interpolations were made by the Lagrangian formula of the 20th order. Preliminary investigation had shown that the application of a still higher order formula would not modify results given to the number of places retained in the standard tables. The accuracy of the final result is checked to some extent by the agreement of the new sums for the equal energy stimulus with the old. To check the results still further, and at the same time to provide other tables of value, the contracted table has been applied to the three standard illuminants of the C.I.E. system. In this calculation these illuminants have been regarded as specified by the alternate entries of the C.I.E. Table V*, which, apart from the extensions beyond 720 mμ., agree with the particulars given in the earlier Table II†. After the products had been found and summed every entry was divided by the sum obtained for the y coordinates, exactly in the same way as was done in preparing Tables VI, VII and VIII‡ for the previous paper. The new

* *Trans. Opt. Soc.* 33, 102 (1931–32). † *Ibid.* p. 77. ‡ *Ibid.* pp. 103, 104, 105.

totals after division and cutting down the number of figures compare with the old totals as shown in table 2:

Table 2

Source	x coordinate		z coordinate	
	Condensed	C.I.E.	Condensed	C.I.E.
Illuminant A	109·8450	109·8472	35·5824	35·5824
Illuminant B	99·0915	99·0930	85·3094	85·3125
Illuminant C	98·0699	98·0705	118·2216	118·2246

These results show that for smooth energy distributions such as these illuminants the coordinates derived from the condensed tables are indistinguishable from those given by the fundamental tables.

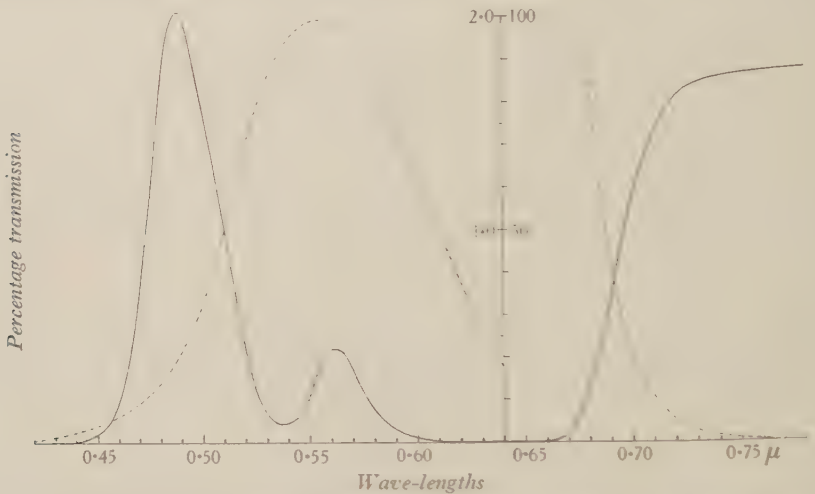

Figure 1. The full curve shows the transmission of a dichroic filter, the ordinate scale for the shorter wave-lengths being fifty times that for the longer wave-lengths. The broken curve represents the standard visibility function, the scale for the shorter wave-lengths being one-fiftieth of that for the longer wave-lengths, thus enabling the products of the ordinates of the two curves to be approximately visualized on a uniform scale for the whole spectrum.

As a further check a computation has been carried out with both tables on a dichroic filter which exemplifies a distribution that might be expected to indicate the presence of faults in certain parts of the new table if it did not indeed happen to illustrate conditions in which these tables could not be safely applied. The results, which are given in the accompanying tables, show that the new procedure agrees well with the old in this example also. The transmissive properties of the filter were measured by Mr H. G. W. Harding and are shown graphically in figure 1. It will be seen that the filter completely absorbs light of the shortest wave-lengths and also that of an intermediate range of wave-lengths. The transmission factors are very different in the two regions where light is passed, as is shown by the scale of ordinates being fifty times as great for one region as for the other. The dotted curve shows the standard visibility function, with the scales in the two parts altered in the inverse ratio to that for the transmission curves. The two taken in con-

junction enable us to see that there is not a great disparity in the light that would be transmitted in the two regions if a moderately high temperature source were used. The small hump in the neighbourhood of wave-length 560 mμ. is of much importance, since it occurs where the visibility of light is greatest. The maximum transmission here is less than one-half of 1 per cent. In spite of this low value good accuracy is obtainable in the measurements with modern equipment. The advances that have been made as compared with pre-war spectrophotometry will be realized from the fact that observations on this part of the curve were made at every 5 mμ. and that the curve has been drawn to pass exactly through every plotted point. On the other part of the curve observations were made at every 5 mμ. up to 700 mμ. and beyond that at every 10 mμ. In the region where the transmission exceeds 80 per cent. two or three of the observed points lie just off the curve by an amount about equivalent to the thickness of the line in the drawing. Owing to the distribution and the values of the transmissions the colour of a source viewed through this filter changes rapidly with the colour temperature of the source. If the slopes of the transmission and visibility curves in the diagrams are considered it will be seen that this filter is likely to provide a good test for the new tables.

It is interesting to consider the contributions of the two regions of transmission separately. This can be done since the transmission factor falls to and remains at zero for a considerable length of the spectrum between the two regions. Taking first the band of shorter wave-lengths, the contributions to the x, y, and z sums were evaluated with the full tables using the measurements made at every 5 mμ., and also with the condensed tables completely neglecting the observations at the odd wave-lengths. The leading figures in the values obtained are as follows:

Table 3

Source	x C.I.E.	x cond.	y C.I.E.	y cond.	z C.I.E.	z cond.
A	10884	10880	24329	24369	19416	19435
B	12966	12959	31614	31674	37978	38002
C	14027	14018	34710	34780	48087	48109

As the observations are recorded to three significant figures, this comparison is very satisfactory.

For the transmission band of longer wave-lengths the observations were partly at 5 mμ. intervals and partly at 10 mμ. intervals. The difference between the two totals was therefore computed by first finding the differences between the measured and the computed intermediate transmissions, the latter being based on the observations at the longer intervals only. The values obtained are as follows:

Table 4

Wave-length	Observed transmission	Computed transmission
655	0·00000	0·00023
665	0·00419	0·00422
675	0·064	0·05866
685	0·218	0·22464
695	0·471	0·46873

From the differences between the last two columns it is found that the differences made by the change to the new tables in this part of the spectrum are of the same small order of magnitude as with the shorter wave-length band.

The totals give the following comparative results:

Table 5

Source		x coordinate	y coordinate	z coordinate	Percentage transmission factor
A	C.I.E.	0·5117	0·3402	0·1480	0·4462
	Cond.	0·5113	0·3404	0·1483	0·4463
	Diff.	− 0·0004	+ 0·0002	+ 0·0002	+ 0·0001
B	C.I.E.	0·3453	0·3446	0·3101	0·4221
	Cond.	0·3448	0·3449	0·3103	0·4225
	Diff.	− 0·0005	+ 0·0003	+ 0·0002	+ 0·0004
C	C.I.E.	0·2862	0·3361	0·3776	0·4280
	Cond.	0·2858	0·3365	0·3777	0·4286
	Diff.	− 0·0005	+ 0·0004	+ 0·0001	+ 0·0005

It need hardly be said that the differences exhibited here are far below those which are perceptible to our eyes. It is instructive to compare them with the large differences of colour coordinates consequent on changing from one source to another. The transmission factor passes through a minimum value as the source increases in colour temperature. This might have been foreseen from an examination of the curves in figure 1. On the other hand the y coordinate passes through a maximum value.

While these examples show that the new tables may often be used with advantage in place of the full standard observer results, it must be emphasized that unlike the full tables they do not specify the standard observer, that they should only be used when it is certain that they will give results in agreement with the full tables, and that when the two differ only the results derived from the full tables have any recognized meaning. The responsibility for deciding when the new tables may properly be used, and labour thereby be saved, must rest in every case upon the experimentalist, who will be guided by the regularity or irregularity of the curve defined by his observations. There is much evidence available to show that the contracted tables may very often be used without sacrificing accuracy, but it is not at the moment possible from existing measurements to provide real typical illustrations of the conditions in which they may not be used. It is of course easy to construct theoretical energy distributions which are not adequately specified by their values at 10mμ. intervals, and for these the new tables would certainly be unsuitable.

Note added February 26, 1934

Since this paper was written Mr Harding has measured the transmission of a filter for isolating the green line from the mercury spectrum. This filter looks a pure yellow by transmitted light although it transmits no spectral yellow. The

observations taken at every 5 mμ. are shown in figure 2, and the freehand continuous curve shows very approximately the transmission values based on these observations. Alternate observations are indicated by large and small dots, and the broken curve represents the transmission values that would be expected when the latter group of observations is disregarded. At the top of the diagram the general character of the errors made by neglecting these observations is shown. Owing to the somewhat irregular shape of the full curve and to the rapidity with which the transmission is changing at some parts of the spectrum these errors are very large in comparison

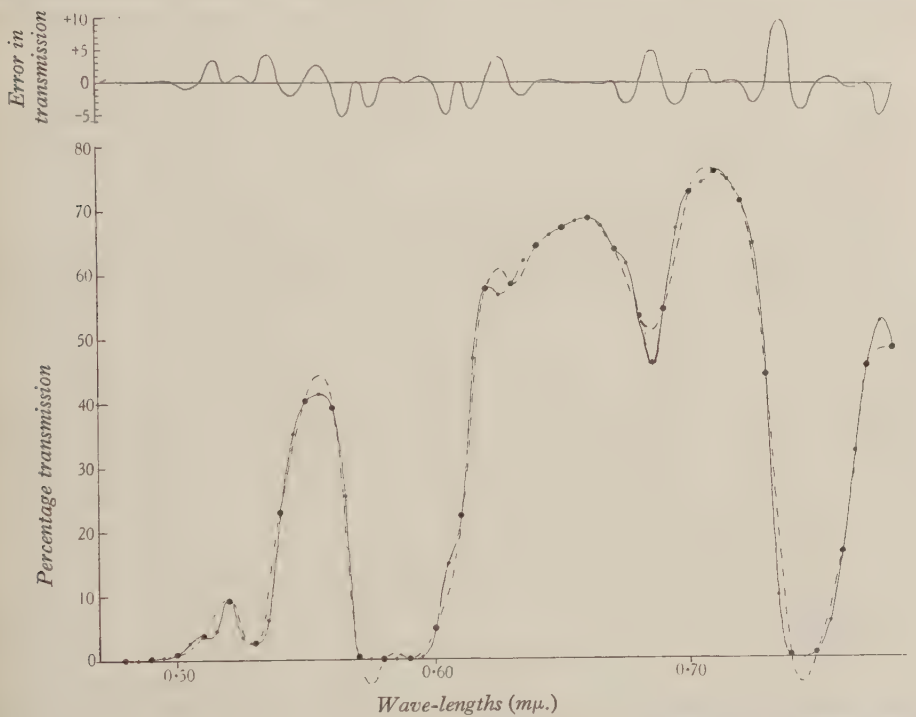

Figure 2. The full curve represents the measured transmission of the filter, and the dotted curve the transmission corresponding to alternate measurements only. The difference between the two curves is shown above. From inspection of this curve too high a value of y (excess of green) and too low a value of x (deficiency of red) is to be expected, and this is borne out by the calculations.

with any possible errors of measurement. With source *B* the C.I.E. colour co-ordinates corresponding to the full curve, and those obtained by using the condensed tables are as shown in table 6.

Table 6

	x co-ordinate	y co-ordinate	z co-ordinate	Percentage trans-mission factor
C.I.E.	0·5520	0·4426	0·0054	18·53
Condensed	0·5491	0·4451	0·0058	18·30
Difference	− 0·0029	+ 0·0025	+ 0·0004	− 0·23

The colour with any reasonably white source is a good yellow, the hue wavelength lying between 586 and 587 mμ. with a white constituent of approximately $1\frac{1}{2}$ per cent. Mr Guild informs me that these differences between the C.I.E. and the condensed results, and also the departure from spectral purity, correspond to differences of colour that are measurable but not evident on inspection. As regards hue this is borne out by the recent investigation on hue-discrimination by Dr Wright and Mr Pitt, who in this region of the spectrum give 1 mμ. as their threshold difference. The relevant spectral colour coordinates of the C.I.E. standard observer are as shown in table 7.

Table 7

λ	x	y	z
586	0·5510	0·4482	0·0008
587	0·5572	0·4421	0·0007
Difference	+ 0·0062	− 0·0061	− 0·0001

In this particular case we may then say that the result given by the condensed tables would be quite satisfactory for ordinary commercial purposes, but not for standardization work.

From the character of the error curve given in figure 2 it is evident that the very fair agreement between the coordinates derived from the two sets of tables is due to the random distribution of the errors made by neglecting alternate observations. It is not of course permissible to assume that in any particular case gross errors such as those exhibited in figure 2 will tend to neutralize one another in the way or to the extent they happen to have done in this example.

DISCUSSION

Mr J. Guild. I think everyone who has at any time to compute colorimetric quantities from energy-distribution data will find it helpful to have these condensed tables in a self-contained and convenient form. The tables being computed so that the sums of the various columns are the same as those of the corresponding columns of the official C.I.E. tables, we have the advantage that quantities evaluated by means of the condensed tables can be directly compared with corresponding quantities computed by means of the full tables without the trouble of reduction to a common denominator. This is particularly useful in the case of the tables for the standard illuminants, as ΣEyt (where t is the transmission or reflection factor of a specimen) gives directly the photometric transmission or the brightness factor of the specimen for the illuminant in question.

There is a statement in the paper which is liable to an interpretation which I am sure the author does not intend. This is the sentence, in the second paragraph, which begins: "The formally correct procedure when this is known. . . ." In his preceding paper he has shown that there is no point in evaluating an integral by summing more terms than there are observed values of either factor in the integrand, whereas the sentence I have quoted may be read as contradicting this and implying

that there are circumstances in which the "formally correct" procedure is, in fact, to interpolate between the measured values. I hope the author will remove any possibility of misapprehension on this point by explaining more fully what he means to convey by "formally correct."

My own view is that if I obtain colour coordinates from energy-distribution data at 10-mμ. intervals, say at 400, 410, 420mμ. etc., I am necessarily obtaining only an approximation to the correct C.I.E. values, and I cannot get (or rather cannot *know* that I get) a better degree of approximation than is obtained by simply adding the products, at these intervals, in accordance with the conclusion in the author's previous paper. If I were to displace my observation wave-lengths by 5mμ., so that they occur at 405, 415, 425mμ. etc., I should, in general, get an appreciably different result from the summation of products, but this difference is unavoidable, being twice the error inherent in either of the separate attempts to obtain the result from incomplete data. I cannot see that there is any way of reducing this uncertainty, the only "formally correct" procedure for obtaining the correct result being in fact to determine the experimental factors at all the tabulated wave-lengths. When we depart from this procedure we do it deliberately, on grounds of economy, when we have reason to believe that the error in the result will not be as great as that which we are prepared to tolerate.

The surprising thing which is shown by the examples selected by the author is that these errors are so small even with absorption curves of extremely peaky and irregular character. The conclusion which I should form on this evidence is that colour-computation from experimental data at 10-mμ. intervals, carried out with these condensed tables, is amply accurate in 95 cases out of 100, but is not good enough, in some cases, for the present standard of accuracy in standardization work. There is, however, a considerable amount of repetition in such work, and the magnitude of the error can be determined by comparing the results obtained from a complete set of measurements with those deduced from a half-set (10-mμ. intervals) for one specimen of any given type. If the error is negligible for the purpose in view, work on all other specimens of the same type can be confined to the larger intervals.

Needless to say, the condensed tables will be of no use to those who imagine that the existence of "standard-observer" tables renders it practicable to evaluate colour coordinates from energy-distribution data to as many significant figures as there are available places on a computing machine. It is sometimes overlooked that the quantities of photometry and colorimetry are physical properties of things, and that *something* has to be measured before we can evaluate them. The best spectrophotometric technique places limits on the significant precision of a colorimetric result computed from energy-distribution, which is ludicrously low compared with that to which significance is attached in some colour calculations. If, however, we are prepared to regard colorimetry as a practical branch of metrology, and not as an arithmetical exercise, these condensed tables are quite accurate enough for a large proportion of the computational work which requires to be done.

Dr W. D. Wright. The condensed tables should be welcomed by colorimetrists both for their own value and as an indication of the trend of opinion on the subject. In standardizing the colour-mixture data for a normal eye, the International Illumination Commission provided tables in which, so far as could be anticipated, interpolation would never be necessary. This meant, on the other hand, that the values were specified at more frequent wave-length intervals than would in general be required, so that the amount of work involved in spectrophotometric calculations appears unnecessarily formidable. The more this work can be reduced the better, and for this reason it would be interesting to know whether the author contemplates the preparation of tables with still larger wave-length intervals. In many cases it should not be necessary to take more than 20 points throughout the spectrum.

It would, I think, add to the value of the paper if the data on the second filter, discussed by the author in the reading of his paper, could be included. The results on the first filter, as given in table 5 on page 378, show that the errors introduced with the condensed tables were quite negligible, and they suggest that even fewer points could have been used. An absorption curve in which the errors become appreciable would therefore be of considerable interest. It would also be of interest to know whether the author has made any calculation of the differences between the values obtained by using the condensed tables and those taken from alternate entries in the original tables.

Finally, I gathered from the author's remarks that the tendency in some countries is to use enlarged, rather than condensed tables. This seems an unnecessary elaboration, and one that would not be favoured by the industrial worker. It would be interesting to know whether this trend originates in the industry itself or in the National Laboratories.

Author's reply. By the formally correct procedure in calculating colour coordinates on the C.I.E. system I mean computation using the full Standard Observer tables with transmission values for the filter considered (or equivalent data for other materials) for intervals of 5 mμ. If it is known, as I have specifically laid down in the passage referred to, that we get accurate results for the properties of the material at the 5 mμ. intervals by interpolating from observations at twice this interval, we naturally interpolate instead of measure, and then proceed as though we had measured everywhere. This is the procedure which I think would have been adopted by everyone at the time the C.I.E. colorimetric system was established, and for the reason that it was the course then contemplated I refer to it as the formally correct procedure.

The preceding paper to which Mr Guild refers shows that we can get the same result under these circumstances with less labour. The condensed tables of this paper are those by which this result can be reached. The alternate entries of the full Standard Observer tables do not yield this result, though they nearly always give a very good approximation to it. Under the conditions we are assuming, the full Standard Observer and the condensed tables give the same result, any differ-

ences in the final figures being due wholly to the limitation in the number of figures retained in each entry of the condensed tables, and these are necessarily of trivial magnitude. I should not under those conditions therefore consider the figures given by the condensed tables as an approximation to the correct result, but as the correct result itself. What was the formally correct procedure is therefore no longer the one which ought to be followed. As soon however as we do not know that observations at 5 mμ. intervals will agree with interpolations from measurements at 10 mμ. intervals the position alters. If measurement at these intervals is all that is available the best result is likely to follow from the use of the condensed tables, but this result will in general be found somewhat in error if observations at every 5 mμ. are subsequently taken and the full C.I.E. tables are used. The examples however suggest that this error will usually be quite small. In any event it is attributable solely to the inadequacy of the spectrophotometric work.

In reply to Dr Wright I have added to the paper particulars of the second filter. I have already pointed out that it is better to use the condensed tables* than alternate entries of the full tables. The differences resulting from using the two alternate sets of entries are known for many coloured materials and are in general quite small. They provided the evidence that tables with larger intervals could often be used without introducing a sensible error. Before preparing the paper I considered still more contracted tables, but the results of partial computation led me to think that they would not prove very useful. An interval of 20 mμ. is certainly too wide to be acceptable for any of the regular work carried out at the National Physical Laboratory, and I should hardly expect to find that anyone would adopt so large an interval after giving it a fair trial over a wide range of colours. It is not unimportant to remember that as the intervals are further increased the saving of labour becomes much less appreciable, while the possible errors increase very rapidly. With a reduction to 20 observations for the whole visible spectrum the additional labour-saving is only 25 per cent, while the outstanding errors are likely in general to be something like five times as great, and in particular cases they may easily be increased ten times or more.

Like Dr Wright I have been puzzled by the elaboration of the C.I.E. tables which has been carried out elsewhere. I have not heard how this movement originated and I do not understand its purpose.

* It would of course be possible to make almost trivial alterations in the Standard Observer tables which would enable alternate entries to be used instead of special tables, but it seems undesirable to alter these agreed figures.

537·745:621·385·3

AN APPARATUS FOR THE OPTICAL PROJECTION OF THERMIONIC VALVE CHARACTERISTICS

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ABSTRACT. A simple and robust apparatus has been developed, primarily for lecture demonstrations, to show the various characteristic curves of thermionic valves on a screen of any desired size. Two mirror oscillographs are employed with axes at right angles, one indicating anode current and the other the grid potential. A suitable alternating voltage of small amplitude from a 50-cycle supply is applied to the grid, causing the characteristic to be traced out 50 times a second, and persistence of vision causes the whole curve to be visible. The instrument indicating current is a Sprenger oscillograph, while the voltage-controlled vibrator is a special instrument designed for the purpose and described in the paper.

§1. INTRODUCTION

The present apparatus has been developed for the purpose of showing to a large audience the (grid-volts, anode-current) or mutual characteristics, and (anode-volts, anode-current) or anode characteristics, and to demonstrate the effect of the variation of the internal and external conditions. In addition to the usual static characteristics, the dynamic characteristics (taken with load in the anode circuit) can also be studied. In this way the difficulties in the accurate matching of reactive loads can be shown.

A mechanical apparatus of this kind possesses advantages in cost, simplicity, and portability over a cathode-ray tube.

§2. MODE OF OPERATION

A beam of light from a gas-filled lamp, pointolite, or arc lamp, according to the size of picture required, is reflected successively by two mirrors which vibrate about horizontal and vertical axes respectively. The first mirror is deflected by a current-controlled oscillograph mechanism connected in the anode circuit of the valve, and its angular displacement at any moment is proportional to the anode current at that moment. After reflection from the current-controlled vibrator, the beam strikes a second mirror attached to the voltage-controlled vibrator and rotating in synchronism with the grid or anode potential as desired. The spot of light on the screen takes up a position on the characteristic curve of the valve, and the whole curve is traced out if the potential is varied continuously. It is convenient to use the

standard 50-cycle supply for this purpose, and a perfectly steady trace without flicker is thus obtained.

Owing to the fact that the beam of light reflected from the first mirror moves through an appreciable angle, the second mirror must have a length of about $1\frac{1}{2}$ in. in a direction parallel to its axis of rotation. No suitable oscillograph could be found to satisfy this condition, and a special indicator, described below, has been designed for this purpose.

If an opaque screen is used, lines can be drawn upon it to represent the axes of the curves. If, however, a ground-glass screen is used, the trace being projected from the side remote from the observers, it is desirable to show the axes as bright lines. The beam of light should then be made slightly broader, so that part is reflected from fixed mirrors alongside the moving ones. The Sprenger oscillograph used as the current-indicating vibrator is already fitted with such a mirror, and one can easily be fitted to the voltage instrument. By this means two perpendicular bright lines are shown on the screen. Their positions can be adjusted to the current and voltage zeros by switching off the vibrators, and turning the appropriate fixed mirrors so as to bring the axis spots into coincidence with the curve spot.

§ 3. THE CURRENT-CONTROLLED VIBRATOR

Any magnetically-operated oscillograph of sufficient sensitivity can be used here, provided that it will operate with its axis of rotation horizontal, but the choice is very much limited by these two conditions as to sensitivity and plane of operation. Sufficient deflection should be produced by a current of a few milli-

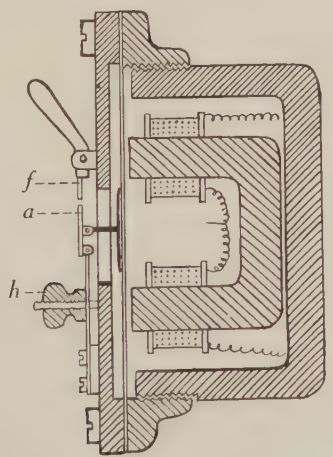

Figure 1. The Sprenger oscillograph. *a*, vibrating mirror; *f*, fixed mirror; *h*, zero-adjustment.

ampères, and a variable shunt should be provided to make it suitable for use with different types of valves.

If the object in view is simply to demonstrate the behaviour of valves in a general manner, and not to examine any particular type of valve, it is possible to obtain

a valve of one of the types giving an anode current of 50 or 100 mA., and this, of course, imposes less severe restrictions on the design of the oscillograph. If, however, the apparatus is required to work with a valve passing only 5 mA. or so, then the only suitable oscillograph known to the author is the Sprenger.

The Sprenger oscillograph, shown diagrammatically in figure 1, is similar to a rather large telephone movement having a resistance of 3000 Ω. Attached to the centre of the diaphragm is a small circular mirror *a* pivoted so that movement of the diaphragm causes the mirror to rotate on its axis. A variable amount of frictional damping can be applied to the mirror, and its equilibrium position is also adjustable by means of a thumb screw *h*. The instrument is usually supplied complete with a small gas-filled lamp of the motor-headlight type, pin-hole screen, and lens. The whole of this optical system is mounted on a frame with the oscillograph instrument and it remains permanently in correct alignment with it. For work with a small opaque screen, or a moderately large translucent ground-glass screen with projection from the rear, the optical system can be used without alteration. When extra brilliancy is required it will be necessary to use a pointolite or arc lamp with suitable optical arrangements to obtain an intense beam.

§ 4. THE VOLTAGE-CONTROLLED VIBRATOR

The usual types of oscillograph have, of course, small light mirrors so that they can readily follow the rapid variations of current without considerable expenditure of power. But for the present purpose a long mirror is necessary in order that the spot of light after deflection at the current oscillograph may not move right off the second mirror. However, a low frequency such as 50 c. sec. may be used, and the

Figure 2. Moving system of voltage-controlled vibrator.

power available can be practically unlimited, so that there is no difficulty in using a mirror 1½ in. long. A moving-iron type of instrument has been constructed. For the present arrangement the moving-iron pattern has obvious advantages in simplicity over the moving-coil pattern. The moving system, shown in figure 2, consists of a strip of glass mirror *A*, ¼ in. wide and 1½ in. long, suspended by a small wire hook from a spiral spring *B*. Cemented to the back of the mirror, near its

lower end, is a strip of stalloy *C*. Two linen threads, of the kind used for sewing, are attached to the outer ends of the strip and pass downwards to adjustable supports. An aluminium damping vane *D*, also cemented to the back of the mirror, dips into a glass tube containing oil. The depth of immersion can be altered by raising or lowering the oil-chamber. The arrangement is thus practically an inverted bifilar suspension. The natural frequency is made high by the application of considerable tension to the spring *B* through the screw and nut. The screw is slotted to prevent it from turning when the nut is rotated.

The stalloy strip is suspended between the poles of a large horseshoe permanent magnet *M* which also tends to keep the moving system in its equilibrium position. The deflecting force is provided by the coil *F* which produces a magnetic field perpendicular to the field produced by the permanent magnet. The strip of stalloy *C* accordingly takes up its position along the resultant magnetic field due to the two perpendicular components. Any desired degree of damping can be obtained by using oil of suitable viscosity, and the depth of immersion of the vane is also variable. The coil can be wound so as to be suitable for direct connexion across the potential-difference applied to the grid or anode circuit, but it is preferable to wind a coil of comparatively few turns, and to use an external series resistance or potential-divider to obtain a suitable amplitude of vibration. In this way a wide range of conditions can be accommodated according to the type of valve, size of screen, etc.

§ 5. (GRID-VOLTS, ANODE-CURRENT) CHARACTERISTICS

The circuit and arrangement of the vibrators is shown diagrammatically in figure 3. The potential of the grid is varied periodically by the application of an

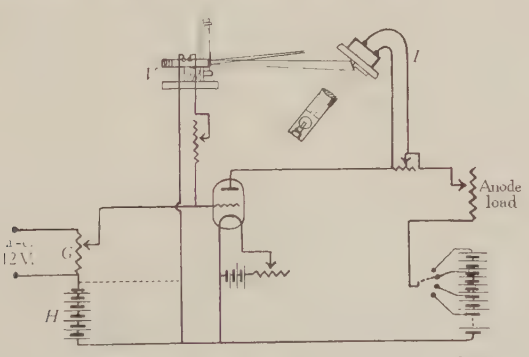

Figure 3. Circuit for (grid-volts, anode-current) curves.

a.-c. supply of about 20 peak volts, adjustable by means of the potential-divider *G*. The battery *H* enables the grid-swing to be varied from 0 to − 40 V. instead of from + 20 to − 20 V., should it be undesirable to show the characteristic to the right of the zero-grid-volts line. The grid potential at any instant is indicated by *V*, the specially constructed instrument, while the corresponding anode current is shown by *I*, the Sprenger oscillograph. Since the steady component of the potential-difference only displaces the mirror of *V* permanently to one side, it is often

possible to transfer the connexion of the vibrator to the upper, positive end of the battery as shown in the dotted connexion. The battery is then not called upon to supply any current, and can be of the usual dry-cell type.

The effect of variation of the filament temperature on the saturation current is at once apparent, and by increasing the anode voltage, the shift of the curve bodily to the left is readily shown.

§ 6. (ANODE-VOLTS, ANODE-CURRENT) CHARACTERISTICS

The Sprenger oscillograph is connected in the anode circuit, whilst the voltage-controlled vibrator in series with an adjustable resistance, is connected across the anode supply, the potential of which is made to vary periodically from 0 to, say, 300 V. by combining a steady potential with an alternating potential having a peak value equal to the steady potential. The connexions are shown diagrammatically in figure 4.

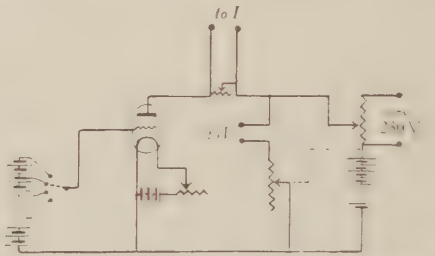

Figure 4. Circuit for (anode-volts, anode-current) curves.

For accurate results the voltage-controlled vibrator should be connected across the combined a.-c. and d.-c. supplies, but as this involves an appreciable strain on the battery supplying the d.-c., it is often better to connect according to the dotted line, across the a.-c. supply only, and to rotate the instrument bodily to give the effect of the steady d.-c. deflection.

If the grid potential is adjusted to various negative values the usual family of curves will be shown, enabling suitable operating points to be found for the valve when used as an amplifier. It is desirable that the grid potential should be changed in steps rather than continuously, so that tappings from the battery should be taken to a rotary switch. The usual precautions are necessary to avoid short-circuiting the intermediate sections of the battery when changing from one tap to the next.

The effects of secondary emission of electrons from the anode can be shown if the grid is maintained at a positive potential rather greater than that of the anode. The negative-resistance properties of the dynatron are shown in this way.

It may be mentioned that by replacing the valve by a fixed impedance the effect of combining two oscillations in perpendicular planes and having various phase relationships can be demonstrated. The well-known straight line, ellipse and circle are produced by altering the relative phases of the currents in the two vibrators.

For demonstration purposes, and perhaps also for routine testing of valves, absolute values of the quantities shown in the curves are not required, but if

necessary the scales can be calibrated. It is seldom, however, that a characteristic has to be determined to a high degree of accuracy.

§ 7. COMPENSATION FOR DISTORTION DUE TO INDUCTANCE OF OSCILLOGRAPH COILS

Owing to the inductances of the windings of the two indicators, there may be a slight phase displacement between the current and voltage, visible on the screen as a double track, and caused by the descending spot not following quite the same path as on its upward movement. This phase displacement may be corrected electrically by introducing capacity into the circuit, generally the anode circuit, in which the current is lagging. The voltage vibrator is made practically non-inductive by the use of a large series resistance.

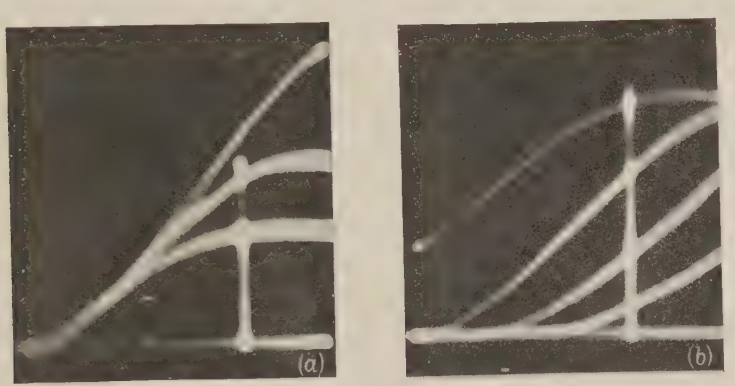

Figure 5. Examples of curves shown by the apparatus. (*a*) (Grid-volts, anode-current) curve with different filament brilliancies, showing effect on saturation current. (*b*) (Grid-volts, anode-current) curve at constant filament voltage and different anode potentials.

When the lag in the anode circuit is only slight, it is simpler to make the adjustment mechanically, by altering slightly the natural frequency of the vibrating system of the voltage-controlled vibrator. This is easily done by turning the milled nut at the top to alter the tension.

The phase relationship between the displacement and the deflecting force in a system vibrating under the influence of a periodic disturbance depends on the frequency of the applied force and the natural frequency of the system. If the latter is altered so as to pass through the resonant frequency, a large phase change occurs. In the present case the natural frequency will be kept well above the applied frequency, but a slight movement in this direction will serve to compensate for the lagging of the current in the anode circuit due to the inductance of the Sprenger oscillograph not being quite negligible.

§ 8. ACKNOWLEDGMENT

The author is indebted to Mr Hellary of the Imperial College workshops for the construction of the vibrator and for suggestions regarding the design of the adjusting devices.

DISCUSSION

Dr D. Owen. The author's work represents a return to the mechanical type of curve-tracer, somewhat surprising in view of recent improvements in and reduction of cost of the cathode-ray oscillograph. It has the advantages, for some purposes, of producing oscillograms on a larger scale and with white light, and these may justify its use. It is important, however, to be assured of reasonably faithful reproduction, and the supply of data in regard to natural frequency and damping, is desirable both in regard to the Sprenger oscillograph and that of the author's own design.

Mr J. Guild suggested that the need for a long second mirror, referred to at the top of page 385, could be obviated by the use of a fixed lens so situated as to produce an image of the first mirror on the surface of the second.

Author's reply. In reply to Dr Owen: The apparatus, in the form described in the paper, is intended for lecture demonstrations. For such purposes an error of even 20 per cent would probably not be very important, the general shape of the curve being all that is required. Actually, however, the fact that the ascending and descending traces coincide within close limits indicates that the curve is faithfully reproduced to approximately 2 per cent. The natural frequency of the Sprenger oscillograph is about 800 c. sec. and the damping is adjustable. The frequency of the voltage-indicating instrument as arranged for operation is approximately 200. This can easily be made much higher, but at that setting compensation is obtained for the lag of the anode current due to the inductance of the anode-current oscillograph. Critical damping can be applied by means of the adjustable oil chamber. In spite of the recent improvements in cathode-ray tubes, it still is not possible to obtain curves of the size and brilliance given, at a small fraction of the cost, by this apparatus without any subsidiary gear.

In reply to Mr Guild: A lens system of the type suggested was given a preliminary trial, but discarded owing to an apparent loss of light. In view of his suggestion it is intended to make another test with more suitable lenses.

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A SIMPLE METHOD OF DEMONSTRATING THE PARAMAGNETISM AND DIAMAGNETISM OF SUBSTANCES IN MAGNETIC FIELDS OF LOW INTENSITY

By Prof. A. O. RANKINE

Received February 16, 1934. *Read March* 2, 1934.

ABSTRACT. The instrument described is the result of an attempt to construct a magnetic gradiometer capable of measuring small distortions of the earth's magnetic field in the same way as the Eötvös torsion balance measures non-uniformities of gravity. Although this purpose has not yet been achieved the first model of the instrument has revealed itself as a means of demonstrating in a convincing manner the paramagnetism or diamagnetism of substances of small susceptibility, for example, the diamagnetism of a few cubic centimetres of water. Moreover, the magnetizing fields employed are much smaller than has hitherto been customary, being of the order of 50 gauss or less. This first model of the instrument is imperfect in certain respects, mainly on account of the use of some brass in the construction. The objections to this material, on account of its ferromagnetic content, were not at first realized, and it will be necessary to construct a new apparatus, employing carefully chosen materials, before full advantage can be taken of the sensitivity available, and demonstration refined to precise measurement.

It seems certain that gases also may be brought within the scope of accurate determination, particularly if access can be had to a suitable laboratory. The present apparatus, however, behaves well enough to warrant description, which is made pending the projected modifications.

The system used also provides a basis for the construction of a new form of very sensitive galvanometer, and this matter is discussed briefly.

§ 1. INTRODUCTION

In 1896 Eötvös* described a magnetic torsion balance which he had constructed for the purpose of measuring spatial non-uniformities of the earth's magnetic field in a way analogous to the operation of his gravity torsion balance. The two instruments are, indeed, described in the same paper. It appears, however, that whereas the gravity balance has since become very widely used in geophysical surveying, the magnetic instrument has produced no important results, if, indeed, it was ever used in the field. The present author began some two years ago to investigate the possibilities of a modification of the Eötvös form of magnetic torsion balance, and mentioned the principal practical difficulty in his presidential address to Section A of the British Association in 1932. It soon became clear that this difficulty, although it prevented the realization of a field magnetic instrument in

* *Ann. d. Phys.* **59**, 375 (1896).

all respects analogous to the gravity balance, did not operate in any substantial manner against the use of the Eötvös principle in a laboratory instrument for measuring magnetic field distortion deliberately caused in the proximity of the apparatus. The result has been the construction of an instrument of an exceptionally sensitive character by which the magnetization of, for example, water in comparatively weak fields of the order of 50 gauss can be readily demonstrated.

The system proves to be virtually that which Kapitza and Webster* rejected as "practically impossible" on account of the smallness of the forces involved. Instead, it is now found to be almost absurdly easy to operate. The surprising thing is that until now no one seems to have realized the magnitude of the force between the pole of a small permanent magnet and a piece of feebly susceptible material really close to the pole, in relation to the delicacy of torsional suspensions about a vertical axis, as opposed to the horizontal axis employed by Kapitza and Webster*.

In the first model of the apparatus now to be described there have been incorporated those suspensions and materials which came readily to hand. The use of ordinary brass in certain constructional elements has proved to be objectionable on account of the ferromagnetic content. As a result, and in a way which is itself of considerable interest, the sensitivity of the instrument is actually reduced to about one-ninth of its nominal value. In spite of the uncertainty as regards precise measurement thus introduced, the system behaves remarkably well, and makes it easy to demonstrate the magnetization in weak fields of slightly susceptible substances. For example, a test-tube of distilled water placed suitably in the apparatus shows a repulsion between it and a magnet pole of about 50 c.g.s. units by a deflection of 5 or 6 cm. of a spot of light on a scale 2 m. distant. Moreover, the balance displays just that immunity from response to large scale temporal fluctuations of the earth's magnetic field (such as diurnal variations and even magnetic storms) which one would anticipate from the theory underlying its operation. Even those erratic fluctuations of artificial origin (usually attributed to the electric currents of railways and trams) which are often so troublesome in magnetic measurements in London laboratories are here ineffective. Only magnetic disturbances of relatively local origin, such as those due to movements of ferromagnetic material in neighbouring rooms, produce any appreciable variation of zero, which is otherwise remarkably constant. It does appear, however, that in order to take full advantage of the sensitivity available, the work must be done in a laboratory in which there can be strict control of any iron or any electric currents. Under these conditions, and with the apparatus reconstructed employing the most suitable materials, calculation provides no reason for doubting that the susceptibilities of gases, even at reduced pressures and in fields of only a few gauss, can be measured with satisfactory precision. As the present apparatus demonstrates the possibilities in a sufficiently striking manner it is considered appropriate to describe its nature and behaviour now, pending the projected modifications of construction and procedure.

* *Proc. R.S.* A 132, 443 (1931).

§ 2. PRINCIPLE OF THE APPARATUS

Consider a system consisting of a horizontal beam (AB, figure 1) of non-ferromagnetic material suspended by a torsional wire or strip CO from a point C vertically above its centre of gravity. From the end A of the beam is suspended, by means of a fibre, a permanent magnet DE with its magnetic axis nearly vertical. (This axis can in fact never be quite vertical, except at the earth's magnetic poles, because the horizontal component of the terrestrial field causes it to tilt slightly.) From the end B of the beam a counterpoise FG of non-ferromagnetic material is suspended in a symmetrical manner, or it may be actually fixed to the beam in a suitable position. The poles of the magnet are approximately at N and S. Let τ be the torsional coefficient of the suspension CO and τ' that of the suspension AD; and let m be the pole strength of the magnet. τ, τ' m

First consider the equilibrium of the system in a spatially uniform magnetic field. The poles of the magnet being precisely equal and opposite in strength, the field gives rise to no translational force on the magnet—only couples. Consequently the remaining forces, of which there are only two—the weight of the magnet and the tension in the fibre AD—constitute a couple. Hence this tension is equal and opposite to the weight, i.e. it is *strictly vertical*. There is therefore no horizontal force on the end A of the beam. This is the essence of the argument. In a uniform magnetic field the system of horizontal forces on the beam consists of two couples only—those due to the twists in the central suspension CO and the subsidiary suspension AD respectively. The existence of these torsions arises from the fact that the magnet has always (except at the earth's magnetic poles) a horizontal component of moment which tends to make the magnetic axis lie always in the magnetic meridian. The control which the earth thus exercises is large compared with the torsional coefficients of suspending fibres of the necessary supporting strength. Accordingly the magnetic axis is always very nearly in the meridian, whatever azimuth the instrument may have, and the torsion in the fibres thus distributes itself between the central and subsidiary suspensions in the proportion $\tau' : \tau$. Thus if τ' is much less than τ, as it may readily be, the effect of the earth's horizontal field on the azimuth of the beam can be made relatively small. It is this effect, however, that prevents the full realization of the magnetic analogue of the gravity gradiometer*, but it is relatively unimportant from our present standpoint. We have generally, in fact, to work from a false zero with some torsion in both suspensions even in a uniform magnetic field. But a uniform field, however strong or inclined to the vertical, can do no more than decrease somewhat the sensitivity of the arrangement as determined by the torsional coefficient of the central suspension alone. The author has shown† that the effective torsional coefficient, i.e. the ratio between any horizontal force-moment applied externally

* The author has discussed this question in detail in his presidential address to the Physical Society, *Proc. Phys. Soc.* **46**, 1 (1934).

† *Loc. cit.*

to the beam and the angle through which it turns in consequence, has a maximum value of $\tau + \tau'$. If only τ', already small, could be reduced to zero the effective torsional coefficient would be simply that of the central suspension itself.

Now suppose that the field is made no longer uniform by the introduction, for example, of a piece of magnetizable material P near the pole S either in front of or behind the plane of the diagram in figure 1 (or the position indicated in almost end-on view in figure 2). This gives rise to a horizontal force of attraction or repulsion perpendicular to the vertical plane containing the beam. This force, acting mutually on the magnet pole and the specimen, causes the suspension AD to be no longer vertical, and the horizontal force is thus transferred to A, and

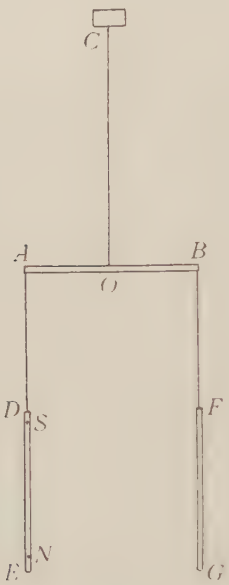

Figure 1.

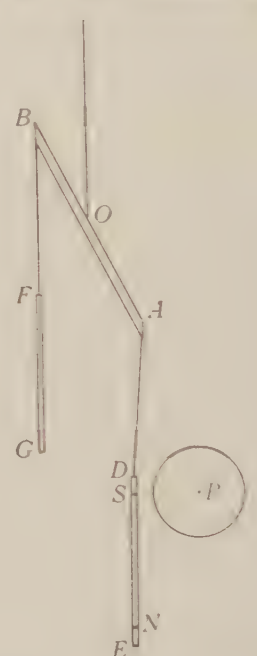

Figure 2. Radius of sphere $P=a$; Distance $PS=c$.

F results in a deflection of the beam until balanced by the torsion. Thus if F is the
θ translational force applied to the magnet and θ is the resulting change of azimuth
b of the beam, the length AO being b,

$$\theta = \frac{Fb}{\tau + \tau'}.$$

The question to be answered is whether the possible values of F, due to the magnetization of the specimen by the magnet pole, and in relation to the torsional coefficients, make it feasible with appropriate optical magnification to measure θ.

§ 3. MAGNITUDE OF THE FORCES OPERATING

Before describing the details of the apparatus it will be convenient to obtain approximate values of the forces of magnetic attraction or repulsion at our disposal. For this purpose we can regard the magnet pole S as situated at a point, and take as our specimen of material a sphere of radius a with its centre at a distance c from the pole (figure 2). It happens that the exact solution of this problem is available, from analogy with the corresponding case in electrostatics. Jeans* gives the formula for the mechanical force between a point charge e and a dielectric sphere as

a, c

$$F = \tfrac{1}{2}\beta e^2 \frac{a^3}{c^3}\left[\frac{1+\beta}{c^2-a^2} + \frac{2c^2}{(c^2-a^2)^2} - \frac{c}{a^3}(1-\beta^2)\left(\frac{a}{c}\right)^{\beta}\int_0^{\frac{a}{c}} \frac{x^{2-\beta}.dx}{1-x^2}\right],$$

where $\beta = \dfrac{K-1}{K+1}$, K being the dielectric constant.

The magnetic equivalent of e is m, the pole strength, and K corresponds to μ, the magnetic permeability. Thus we substitute for β the value

$$\frac{\mu-1}{\mu+1} = \frac{4\pi\kappa}{2+4\pi\kappa} = \frac{2\pi\kappa}{1+2\pi\kappa},$$

where κ is the magnetic susceptibility.

κ

If we concern ourselves with values of κ small compared with unity, i.e. if we deal simply with paramagnetic and diamagnetic substances, the formula becomes simplified and the integration contained in it is performable. The author is unaware of this adaptation of the formula having been given previously and accordingly presents it.

With κ small as indicated, β becomes $2\pi\kappa$, and, being thus itself small, can be neglected in the formula except as the multiplier of the whole expression. We obtain

$$F = \pi\kappa m^2 \frac{a^3}{c^3}\left[\frac{1}{c^2-a^2} + \frac{2c^2}{(c^2-a^2)^2} - \frac{c}{a^3}\int_0^{\frac{a}{c}} \frac{x^2dx}{1-x^2}\right].$$

Writing $a/c = s$ (which < 1 necessarily) integration gives

s

$$F = \frac{\pi\kappa m^2}{c^2}\left[\frac{s(1+s^2)}{(1-s^2)^2} - \tfrac{1}{2}\log\frac{1+s}{1-s}\right].$$

This is conveniently transformed by the substitution $\log\dfrac{1+s}{1-s} = y$, which corresponds to $s = a/c = \tanh\frac{1}{2}y$.

The formula then reduces to

$$F = \frac{\pi\kappa m^2}{4c^2}(\sinh 2y - 2y).$$

For convenience of calculation of F the values of the function in brackets have been given in table 1.

For intermediate values of $\sinh 2y - 2y$ in relation to a/c a continuous curve can, of course, be drawn.

We may take, for example, a case which will be involved later, and assign the

* J. H. Jeans, *Electricity and Magnetism*, 1920 edition, p. 290.

Table 1.

y	$\frac{a}{c} = s = \tanh \frac{y}{2}$	$\sinh 2y - 2y$
0	0	0
0·2	0·0997	0·011
0·4	0·1974	0·088
0·5	0·2449	0·175
0·6	0·2913	0·309
0·8	0·3800	0·776
1·0	0·4621	1·627
1·2	0·5370	3·066
1·4	0·6044	5·392
1·6	0·6641	9·046
1·8	0·7163	14·686
2·0	0·7616	23·290
2·4	0·8336	55·951
3·0	0·9052	195·713

following values: $c = 1{\cdot}80$ cm., $a = 1{\cdot}28$ cm. and $m = 50$ c.g.s. units. The corresponding value of a/c is 0·711, and $\sinh 2y - 2y$ equals 14 approximately. Thus

$$F = \frac{\pi \times 2500 \times 14}{4 \times 1{\cdot}8^2} \cdot \kappa = 8{\cdot}48 \times 10^3 \kappa.$$

The maximum value of the field stimulating the material of the sphere is $50/0{\cdot}52^2 = 180$ gauss approximately, and it falls off rapidly towards the remote side of the sphere. Assuming that κ has in such relatively weak fields the same value as that measured in fields of some thousands of gauss—an assumption needing verification and one which the present system promises to elucidate—we may anticipate that for water F will have the value $-7 \times 8{\cdot}48 \times 10^{-4}$ dyne, or will constitute a repulsion of $5{\cdot}9 \times 10^{-3}$ dyne approximately. For air, taking $\kappa = 3{\cdot}2 \times 10^{-5}$, the corresponding force, this time an attraction, will be $2{\cdot}7 \times 10^{-4}$ dyne. The detection of such apparently small forces is quite easy, as will be seen, if we take advantage of the delicacy of available torsional fibres suspended vertically.

§ 4. DESCRIPTION OF THE INSTRUMENT

In the instrument first made and now to be described full advantage has not been taken of the sensitivity actually attainable. The components have been chosen on the basis of immediate availability and can certainly be improved greatly, given the necessary time. The main suspension (a, figure 3), for example, is the metal strip which was previously incorporated in one of the gravity gradiometers used by the Imperial Geophysical Experimental Survey for gravity work in Australia during 1929–30. Its length is about 30 cm. and its torsional coefficient approximately 0·16 dyne-cm. per radian. It is much stouter than it need be for present purposes, as it would normally be used to support 100 gm. or more, whereas the

total weight now supported is only 17 gm. According to the data available this smaller weight could be supported satisfactorily by a suspension having a much smaller torsional coefficient of about 0·02 dyne-cm. per radian.

This suspension supports a beam *b* in the form of a right-angled triangle of sheet aluminium of thickness 25 s.w.g., the length of the hypotenuse side being

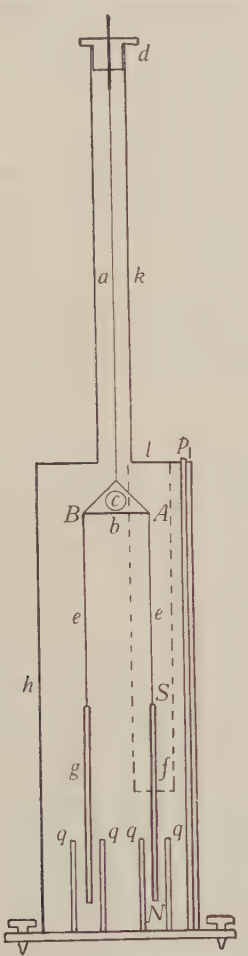

Figure 3. This diagram is roughly to scale. The two other pillars p_2 and p_3 are not shown. The tube *l*, shown dotted, is in front of the magnet.

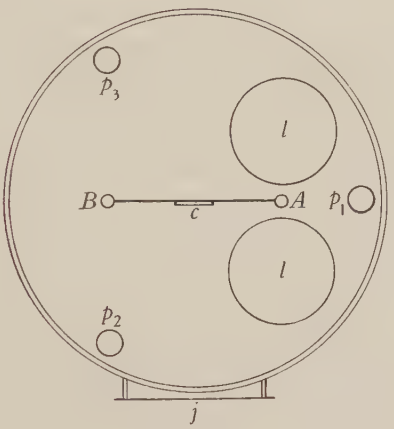

Figure 4. The poles of the suspended magnet are below the point *A*, and the torsion head above *c*. This diagram is on a larger scale than that in figure 3.

5 cm. and the weight approximately 1 gm. It carries the galvanometer mirror *c* for optical observations. The metal strip is fastened to the torsion head *d* and to the beam at *C* and *O* respectively by the small screw clamps supplied with the strip itself by the makers. At the points *A* and *B*, which are symmetrically disposed at approximately 2·3 cm. on opposite sides of the main axis of suspension these are supported by silk fibres *e*, the magnet *f* and the counterpoise *g* respectively. The

silk fibres are about 17 cm. long and have torsional coefficients of about 0·01 dyne-cm. per radian, which is small compared with that of the central suspension. They carry at their lower ends small hollow cylinders of brass into which are inserted the magnet and the counterpoise, friction giving adequate support. The magnet is of Darwin's 15 per cent cobalt steel $\frac{1}{8}$ in. in diameter and 5 in. long, and its magnetic moment of about 570 c.g.s. units has remained remarkably constant during the two years it has been in the author's possession. Its pole strength is estimated to be roughly 50 c.g.s. units. The counterpoise is a brass rod also $\frac{1}{8}$ in. in diameter, and, like the magnet, weighs approximately 8 gm.

In order to protect the suspended system against draughts and other air convection currents the whole is enclosed by a sliding vertical brass cylinder *h*, provided with a glass window *j* suitably placed to allow optical observation. The base, fitted with brass levelling screws is also of brass, and the upper part of the enclosure, including the brass tube *k* surrounding the central suspension and brass torsion head, is supported upon the base by three vertical pillars p_1, p_2 and p_3 at 120° apart as shown in figure 4. These are of $\frac{1}{4}$-in. brass rod, and play an interesting part in the behaviour of the apparatus, as will be seen later.

To allow specimens of various substances to be brought into the necessary close proximity to the magnet poles two metal tubes *l*, closed at the bottom and open at the top, project vertically into the interior of the case. Into these the specimens can be lowered. One could, of course, employ the forces due to *both* poles, since the force depends upon m^2. I have preferred, however, to use the lower pole *N* to secure electromagnetic damping by causing it to move in the annular space between two copper cylinders (*q*) $\frac{1}{8}$ in. thick. It is easy to adjust the positions of the cylinders so that the suspended system is critically damped. Accordingly the closed bottoms of the tubes *l* reach only a few centimetres below the upper magnet pole *S*, and the effect on only one pole is used for purposes of deflection. The tubes themselves were originally of brass of internal diameter 2·8 cm., and of thickness 26 s.w.g.; their axes were about 3·7 cm. apart, so that the axis of the suspended magnet in the symmetrical position was about 4 mm. from the outer surface of each cylinder. These surfaces served the purpose of stops, limiting the angular amplitude of the suspended system to between 5 and 6 degrees. It was quickly discovered, however, that these brass tubes exercised a large magnetic control, presumably on account of their content of iron. The magnet, on approaching either tube, gathered speed and eventually remained in contact with the tube, unless it rebounded and stuck to the opposite tube instead. The substitution of specially made copper tubes of nearly the same dimensions surmounted this difficulty, and a stable zero was secured.

The vertical brass pillar p_1 produced an unexpected effect, greatly diminishing the sensitivity of the arrangement. It was observed that if a second brass rod (like the counterpoise) was substituted for the magnet, the period of oscillation of the system about the central axis was about 165 sec. This became reduced to about 55 sec. with the magnet in place. A considerable control, additional to that of the central suspension, was implied, the total control being nine times that of the sus-

pension, since the periods observed are in the proportion 3 : 1. This extra control was traced to the attraction between the brass pillar p_1 and the magnet, affecting in this case both poles. In the symmetrical position the pillar is about 1 in. from the magnet, yet in a deflected position a component of the attraction, even at this distance, provides a substantial restoring force. This action was verified by later observations in which it appeared that the beam had two other positions of stability, namely, when the vertical plane through it passed nearly through one or other of the remaining pillars p_2 and p_3, even though 120° of torsion existed in the suspension.

The brass pillars still remain in the instrument as it serves purposes of demonstration well, and, indeed, it is not considered worth while to secure the additional sensitivity clearly available (in the ratio 9 : 1) by substituting iron-free pillars, until laboratory conditions can be properly controlled as regards movements of bodies containing iron. Moreover, it seems desirable to construct an entirely new apparatus employing materials chosen with scrupulous care in respect of magnetic properties*. At the same time it will be possible to incorporate the finer suspension mentioned earlier, and to make provision for the essential dimensions to be accurately known, thus enabling precise measurement to take the place of mere demonstration.

At present the optical magnification used is about 3800, that is, the mirror on the beam is about 190 cm. from the scale over which the light-spot moves, and 3800 mm. on the scale thus correspond to a rotation of the beam through one radian. It is probable that this magnification can be increased with advantage.

§ 5. BEHAVIOUR OF THE INSTRUMENT

As already indicated the dimensions and other constants of the apparatus are known only approximately, and cannot be determined conveniently with precision. Accordingly it is proposed now only to indicate how the apparatus behaves under roughly specified conditions. With regard to its sensitivity the introduction into one of the copper containers of a sufficiently long loosely fitting test-tube containing a few cubic centimetres of distilled water produces a deflection of about 40 mm. when the original position of the beam is approximately symmetrical. Of this about 15 mm. is attributable to the test-tube, the glass of which happens to be, like water, diamagnetic. When the beam is originally nearer the used copper container than the other the repulsion is much larger, as the force varies very rapidly with proximity. With a piece of bismuth moulded into the shape of the test-tube the repulsion is so large as to make the beam reach its limits of free movement, corresponding on either side of the central position to about 400 mm. A piece of the same $\frac{1}{4}$-in. brass rod as was used to construct the pillars (p) displays still greater attraction. A further interesting observation is that a cylinder of aluminium on being lowered into the copper tube first repels the magnet suddenly, and thereafter slowly exhibits the attraction appropriate to its paramagnetism. The

* The cadmium-copper recommended by Kapitza and Webster (*loc. cit.*) may prove to be suitable.

explanation of the initial repulsion is the development of eddy currents in the aluminium while it is in relative motion with reference to the magnetic field of the suspended magnet. These are some examples of the efficacy on the instrument.

On the question of its behaviour as regards constancy of zero it is observed to be extremely sensitive to changes of ferromagnetic structure in its immediate neighbourhood. Any things containing iron carried on one's person make themselves very evident on approach to the instrument, and even near-by articles, such as chairs, if they contain iron nails, have to be kept in a fixed position if the zero is not to change appreciably. Still more is it necessary to avoid movements, even in neighbouring rooms, of iron objects such as retort stands, and to suppress large slow changes of current in inductive circuits.

The evidence so far obtained is, however, that if no such magnetic changes occur the mere fixed presence of iron or of constant direct currents in the neighbourhood is not objectionable. (Rapidly alternating currents are, of course, always ineffective having regard to the long period of the suspension.) Observations in the Imperial College laboratory on a Sunday morning, for example, showed a zero constant to within 1 mm., proving that temporal changes of magnetic field of a character uniform enough in space, find the instrument irresponsive, as, indeed, theory predicts. This refers in practice not merely to the diurnal variations of the terrestrial magnetic field and magnetic storms, but to those still widespread enough magnetic fluctuations due to the District Railway at South Kensington. One finds no evidence of those spasmodic fluctuations so noticeable in the use of that relatively insensitive instrument, the Kew magnetometer. It seems reasonably certain that given rigid control over artificial magnetic changes in the immediate neighbourhood, and with proper design of the modified apparatus, we can anticipate a zero constancy comparable with that displayed by the Eötvös gravity instrument, while at the same time taking advantage of the full sensitivity available, probably 100 times that now used. This will imply the feasibility of employing a weaker magnet in the suspension, so that the relation between field and intensity of magnetization can be determined down to fields of a few gauss, even in the cases of gases.

§ 6. ADVANTAGES OF THE SYSTEM

Kapitza and Webster* have already discussed in some detail the advantages of a system in which the magnet, instead of the specimen as in the Curie balance, is suspended. These apply in the main also to the present arrangement. The practicability of making the measurement a null one is very important, as it surmounts the difficulty of the precise measurement of the small and variable separation distances which is necessary in the deflection method. It would clearly be easy to incorporate in the present apparatus also suitably disposed coils of wire, so that by the passage of a certain readily measurable current the magnetic effect of the specimen under investigation could be balanced.

The control of the physical conditions of the specimen, as regards temperature,

* *Loc. cit.*

for example, is also within the scope of experiment; and, although the measurements would be in the main comparative, an approximation to absolute determinations would evidently be possible. For example, experiments made with the present imperfect instrument showed that when a volume of water approximately spherical and of radius about 1·28 cm. was placed with its centre about 1·80 cm. from the rough location of the magnet pole of 50 c.g.s. units, a deflection of 35 mm. was obtained. (The thin glass container showed a negligible effect.) This implies a force of repulsion of approximately $5{\cdot}77 \times 10^{-3}$ dyne, taking the effective torsional coefficient to be nine times that of the main suspension, i.e. 1·44 dyne-cm. per radian, the half length of the beam 2·3 cm., and the optical magnification 3800 mm. per radian. Equating this to the force calculated for the above conditions on p. 496, we obtain

$$8{\cdot}48 \times 10^{3}\kappa = -\,5{\cdot}77 \times 10^{-3}$$

or

$$\kappa = -\,6{\cdot}8 \times 10^{-7} \text{ e.m.u.}$$

No stress is laid here on the fact that this value is near the usually accepted value of $-\,7 \times 10^{-7}$ e.m.u. for intense magnetic fields. There is too much uncertainty in the present apparatus as to the precise values of the small dimensions involved. The location of the magnet pole is also not known accurately, and the assumption of its being a point pole is surely too bold. Moreover, it remains yet to be proved or otherwise that the susceptibility is the same in strong and weak fields. All that can be affirmed now is that it is of the same order.

§ 7. FURTHER USE OF THE APPARATUS

Besides being useful for demonstration purposes the present comparatively insensitive instrument will serve as a means of controlling the components to be used in later models of greater accuracy and sensitivity. It will enable exacting tests to be made of the magnetic properties of all materials proposed to be incorporated, and thus ensure freedom of the suspended system from spurious control such as has become evident in the first example.

§ 8. APPLICATION TO CURRENT-MEASUREMENT

The practicability of using current control in these sensitive measurements of magnetic field distortion immediately suggests that the system under discussion provides a basis for the construction of a new form of galvanometer capable of measuring extremely small currents. It is to be noted that the use of only one magnet secures complete astaticism, since no matching of different magnets is involved. The suggested galvanometer would be of the moving magnet type, and it would be convenient to modify the suspension described earlier by making horizontal (instead of nearly vertical) the axis of the suspended magnet, and fixing the counterpoise to the appropriate position on the other side of the beam. The theory as regards behaviour in a uniform field and immunity from large-scale

temporal changes of field is thus in no way modified. The normal position of the beam would be chosen to be as nearly as possible at right angles to the magnetic meridian, since the magnet, preferably shorter than 5 in., say 2 in., would remain always with its magnetic axis along the meridian. The arrangement would be as in figure 5. Two coils or solenoids with axes coincident with that of the magnet would be disposed as shown, as closely as freedom of movement of the magnet would permit. The sense of winding would be such that the magnetic forces on the two poles due to the current in the solenoids would conspire to turn the beam in the same direction. We may see from the following calculation what may be

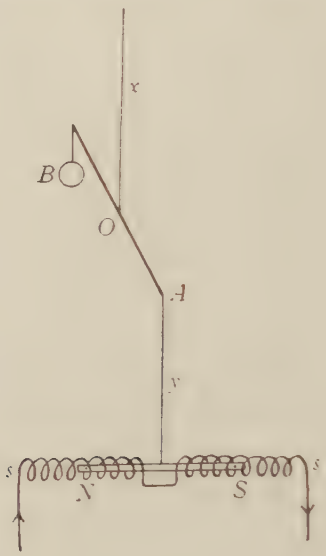

Figure 5. *NS* magnet, *B* counterpoise, *ss* current-bearing solenoids, $OA=b$, *x* main suspension, *y* subsidiary suspension.

anticipated in the matter of sensitivity. The force on each pole is $\frac{4}{10}\pi nmC$, where C is the current in amperes, and n the number of turns per cm. on the solenoids. This force, doubled, at the end of an arm of length b, gives a force moment $\frac{4}{5}\pi n\pi bC$, and if θ is the twist imparted to the central suspension, and τ the effective torsional coefficient, equilibrium corresponds to

$$\tau\theta = \tfrac{4}{5}\pi n\pi bC.$$

If M is the optical magnification, and d the scale deflection $\theta = d/M$, so that

$$C = 5\tau d/4\pi nmbM.$$

As an example of practically possible values we may take the magnet to be 2 in. of $\frac{1}{8}$ in. diameter 15 per cent cobalt steel, which will weigh a little over 3 gm., and provide poles about 50 c.g.s. units approximately 4 cm. apart. (This leaves ample room for the solenoids in which it would be desirable for the magnet poles normally to occupy the centres.) A value of τ of 0·02 dyne-cm. per radian would be

considerably larger than necessary, but we assume this. Finally, with $b = 2$ cm., say, $n = 10$ turns per cm., and $M = 4000$ mm. per radian, we obtain

$$\frac{C}{d} = \frac{5 \times 0{\cdot}02}{4\pi \times 10 \times 50 \times 2 \times 4000} \text{ A./mm.}$$

$$= 2 \times 10^{-9} \text{ A./mm. approximately.}$$

This gives the anticipated sensitivity in a galvanometer which, by reason of the small number of turns of copper wire assumed, would have a negligible resistance. With n sufficiently multiplied by using finer wire, a sensitivity of 100 or perhaps even 1000 times as great could presumably be obtained.

Such a galvanometer would have, however, an unusually long time period—in the above case about 220 sec.—so that it could be applied only to the detection and measurement of very slowly varying currents. As in all galvanometers quick action implies sacrifice of sensitivity. By making $b = 2$ mm., instead of 2 cm., for example, the time period would be reduced to 22 sec., while the sensitivity would also be reduced in the same proportion of ten to one.

DISCUSSION

Dr H. Shaw. It is interesting to note that although the Eötvös torsion balance, and the gravity gradiometer which suggested this apparatus to the author, are essentially field instruments, the new instrument, at any rate for the present, appears to be of the laboratory type. I am also interested to hear of the difficulties which have been encountered as a result of the magnetic nature of brass, and which bear out my own experience. On occasions I have found the magnetism of the brass to be sufficiently large to produce effects which completely masked its gravitational effect.

I should like to ask the author whether he has yet made any quantitative measurements.

Dr L. F. Bates. The author has shown us a delightfully simple piece of apparatus and I am sure that we all look forward to hearing more concerning the behaviour of substances in weak magnetic fields. We have no right to assume that the susceptibility is a constant until it has been proved to be so for these low fields. One of the most remarkable things about the apparatus is that a vertically suspended magnet, which must have some horizontal component of its magnetic moment, can behave so well. Indeed, I am surprised that the author has not needed to trouble about this horizontal component. If he should need to do so with the new form of the apparatus, I commend to him the "gallows" which Chattock and I used for getting rid of such a component some years ago.

The question of electromagnetic control is an intriguing one. I note that the author speaks of a system of coils for this purpose. Actually, I think he will find when he settles down to this particular problem that he will obtain all the control required, simply by using a single long straight wire placed parallel to the cross-piece of the torsion balance and at a little distance below the magnet, but not

vertically below it. The lines of force will be circles concentric with the wire and, obviously, the lower pole will be acted upon by a greater force than the upper one.

Dr L. Hartshorn said that for the purpose of absolute measurements it was necessary to know the permeability of the substances of which coil formers are made. It had been found that both marble and pyrex are diamagnetic, to an extent which could not be neglected in precision work, and it is probable that the author's instrument would be of considerable value in this direction, since it appears to be more sensitive than existing methods and also to be applicable to finished coils.

Dr F. D. Smith. The magnetic torsion balance described in this paper appears to have great possibilities in its present form and also to afford scope for improvement in the future. It is interesting and profitable to examine the conditions that make for an even better performance—that is, increase of sensitiveness with reduction of periodic time.

θ, F The formula for the deflection θ in terms of the force F acting on the pole is

$$\theta = Fb/\tau \qquad \text{.....(1),}$$

b, τ, ω where b is the arm of the balance and τ is the stiffness of the suspension: τ' is assumed to be small in comparison with τ. The resonance angular velocity ω is given by

$$\omega^2 = \tau/I \qquad \text{.....(2),}$$

I where I is the moment of inertia of the suspended system. These equations give

$$\theta\omega^2 = Fb/I \qquad \text{.....(3),}$$

where $\theta\omega^2$ is a factor of merit since increase of θ and ω are advantageous. In fact, the design should be such that $\theta\omega^2$ is a maximum. Let us see what factors contribute to increase this product.

In cases where the magnet pole is near the sphere, that is, where s (or a c) is nearly unity,

$$\sinh 2y - 2y \fallingdotseq 2c^2/(c-a)^2 \qquad \text{.....(4),}$$

whence

$$F = \frac{1}{4}\cdot\frac{\mu-\mu'}{\mu+\mu'}\cdot\frac{m^2}{\mu'}\cdot\frac{1}{(c-a)^2} \qquad \text{.....(5),}$$

m, μ, μ' where m is the strength of the magnet pole, μ is the permeability of the substance under examination and μ' is the permeability of air; $(c-a)$ is the distance between the pole and the surface of the sphere. This expression, which is derived from the formula given by the author, can be identified as the formula for the force of attraction between the pole and a thick extensive plane slab of material at distance $(c-a)$ and having permeability μ. It seems that the shape of the body under examination does not matter since the same formula applies for a sphere and a slab provided always that the pole is close to them. In fact, only the material in the immediate vicinity of the pole contributes materially to the mechanical force.

The author uses 15 per cent cobalt steel for his magnet, the ratio of length to diameter, on which the demagnetizing effect of the poles at the ends depends,

being 40 to 1. A lighter magnet results from the use of steel of better quality, as the following table indicates.

Steel	Coereive force	Length/ diameter
15 per cent cobalt steel	190	40
35 per cent cobalt steel	240	32
M. K. aluminium steel	700	11

It has been assumed, as a rough approximation, that the ratio n of length to diameter varies inversely as the coercive force for a given intensity of magnetization I_r. The new Japanese magnet steel, now available in England, enables the mass of the suspended system to be cut down to one quarter, the periodic time being thereby halved. If the radius of the circular section of the magnet is r, the length is $2nr$ and the formula for the moment of inertia of the suspended system becomes

$$I = 4\pi\rho n r^3 b^2 \qquad \text{.....(6),}$$

where ρ is the density of the magnet.

Writing $m = \pi r^2 I_r$ we reduce equation (3) to

$$\theta\omega^2 = \frac{\pi}{16}\cdot\frac{\mu-\mu'}{\mu+\mu'}\cdot\frac{I_r^2}{\rho n}\cdot\frac{r^2}{(c-a)^2}\cdot\frac{1}{br} \qquad \text{.....(7).}$$

We wish to make $\theta\omega^2$ a maximum. The term $\frac{\mu-\mu'}{\mu+\mu'}\cdot\frac{1}{\mu'}$ represents the magnetic quality of the substance under test and is not under control. The term $I_r^2/\rho n$ represents the magnetic quality of the magnet; since it must be as large as possible, n must be small, that is, the magnet must be short and thick; the density must also be small and, though the actual density of the magnet steel is not under control, it is possible to reduce the effective density by making the magnet of dumb-bell shape with enlarged ends. The term $r^2/(c-a)^2$ is a measure of the proximity of the magnet to the substance under test; since r must be less than $(c-a)$, it has a maximum value of unity. The final term $1/br$ determines the scale of size of the balance; to make this term large, the arm of the balance b and the radius of the magnet cross-section r must be small. It seems that the magnetic torsion balance should be made as small as possible, the practical limit being set by the fineness and permissible length of the suspending wires. This conclusion is in accordance with the historical development of other electrical instruments, for example the quadrant electrometer and its minute modern counterpart, the Lindemann electrometer.

On the basis of these calculations I venture to predict that the magnetic torsion balance may be greatly reduced in size with consequent increase of sensitiveness or quickness of response, or both.

Mr J. E. Calthrop. I gather that the suggested galvanometer would have a very long period. One of its possible uses might therefore be to demonstrate the existence of very slow electric oscillations of a period of a few seconds. Such an

experiment has been described by Prof. R. W. Pohl of Göttingen. The oscillations were produced by allowing a charged condenser to discharge through a very large inductance, and the period of the oscillations was observed by means of a galvanometer of special design and having a period of about 35 sec. For instructional purposes such an ocular demonstration would be of great value.

Dr A. B. Wood. The direct applications of the instrument to the measurement in the laboratory of the magnetic properties of materials make it extremely valuable, apart altogether from its possible application as a field instrument, for which purpose it was originally designed.

In his theoretical treatment on page 395 the author considers the forces acting on a single pole of the suspended magnet. What is the resultant effect of the two opposite poles? How is the sensitivity affected by the distance apart of the poles, and by the length of the magnet relative to the size of the specimen?

If further sensitivity is at any time required, it seems possible that the new Japanese aluminium steel, supplied by Lucas of Birmingham, may prove superior to cobalt steel for the suspended magnet, having a considerably greater coercive force for a given volume.

The demonstration gave me the impression that extremely few materials may be regarded as non-magnetic and we shall look forward to a further paper by the President in which he reveals the non-magnetic substance to which he referred so mysteriously during the demonstration.

Author's reply. It will be convenient if I reply first to Dr Wood's questions, as the answers have a bearing on some comments I have to make on Dr F. D. Smith's contribution to the discussion. If the magnet is long in comparison with its distance from the specimen, itself also supposed relatively small, the pole remote from the specimen experiences a negligible force, since something like an inverse fifth-power law operates. But if the specimen is large –if, for example it extends parallel to the magnet over its whole length –the resultant effect would clearly be twice that due to a single pole. The forces also depend on the shape of the specimen, although if it is very close to the magnet pole the shape is relatively unimportant, as Dr F. D. Smith has pointed out.

As I have already indicated I have preferred to use the lower pole for electromagnetic damping, and not to procure increased sensitivity. This implies a practical limit to the extent to which the magnet may be shortened. Moreover, although I have not calculated precisely the forces operating when the inter-polar distance is diminished, the problem being too difficult, I am convinced that ultimately such shortening will decrease the resultant force, because of the tendency for more of the lines of force to pass directly from pole to pole without penetrating the specimen. At the same time I believe that some reduction of length would be advantageous in reducing the period in the manner which Dr F. D. Smith suggests.

I am glad to have from both Dr Wood and Dr F. D. Smith the information regarding the properties of the new aluminium steel. Its greater coercive force will permit the reduction of length contemplated for the magnet, and if its greater

coercive force implies—I am not sure of this—that the remanent magnetism will be greater also, the new steel will afford still greater advantages.

I welcome Dr F. D. Smith's clear exposition of the questions of sensitivity and quickness of response, which in the present model are not as good as they might be. The instrument was in the first instance designed as a means of measuring gradients of the earth's field, and in its present adaptation to the measurement of susceptibility somewhat different conditions apply, so that suitable modifications must be made in new designs. I think Dr Smith's general conclusions are correct, and correspond not only to those for the electrical instruments to which he refers, but also to those reached by Boys in relation to his gravity balance to which my instrument is more nearly akin. But I am not sure how far it will be possible in practice to adhere to the theoretical argument, which assumes that it will always be possible to obtain materials of the properties and dimensions demanded by the theory. There will be, as Dr Smith anticipates, a practical limit to the fineness of the suspending wires, determined by the skill of the construction and the delicacy of handling on the part of the user. I expect also that spurious movements due to air convection will become more troublesome with very light suspended systems. But there is clearly much room for improvement on the lines Dr Smith has suggested, especially with regard to quick action, since the sensitivity already seems to be ample.

In reply to Dr Shaw: I have made some rough comparisons which show that substances have approximately the same relative susceptibilities in weak as in strong fields. I anticipate being able to make the measurements more precise when the null method indicated is employed. In this connexion I am much indebted to Dr Bates for his suggestion that a single straight current-bearing wire outside the case of the apparatus will serve. I have since tried this, and it works admirably with currents of the order of milliamperes. I place the wire under the apparatus in the vertical plane containing the beam. The wire is in fact about 6 cm. vertically below the lower end of the magnet. I do not understand why Dr Bates says it should not be quite vertically below it. As at present arranged it produces a horizontal force on the pole perpendicular to the plane of the beam, which is what we require.

Dr Bates' other suggestion also interests me very much, but it applies more to the original attempt to design a magnetic gradiometer which I described in my presidential address. In the present use of the instrument I actually prefer the magnet to have a definite horizontal component of moment, so that it will take up a nearly constant azimuth. When I have time to revert to the first project I shall have to examine whether the Chattock and Bates "gallows" will fulfil sufficiently the extraordinarily stringent demands for the elimination of control by the earth's horizontal field.

621.396.11

THE APPLETON-HARTREE FORMULA AND DISPERSION CURVES FOR THE PROPAGATION OF ELECTROMAGNETIC WAVES THROUGH AN IONIZED MEDIUM IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD. PART 2: CURVES WITH COLLISIONAL FRICTION

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(from the National Physical Laboratory)

Communicated by R. A. Watson Watt, January 19, 1934. *Read March* 2, 1934.

ABSTRACT. This paper presents curves given by the Appleton-Hartree formula and illustrating the effect of collisional friction on the indices of refraction and attenuation for the propagation of electromagnetic waves through an ionized medium in the presence of an external magnetic field. Four typical frequencies have been chosen for the calculations, one from each of the classes into which the frequencies fall when collisional friction is absent, as described in part 1. The corresponding wave-lengths are 80, 240, 400 and 1000 m. The various stages in the effect of increasing collisional friction have been found to be usefully represented by collisional frequencies of 10^5, 10^6, 10^7 c. sec. and curves are given showing the indices of refraction μ_r and attenuation κ_r, $(r = a, b)$, the real part and the imaginary part of M_r^2 or $(\mu_r - i\kappa_r c/p)^2$, and the polarizations of the basic modes as functions of the electronic density for each of the four frequencies and collision frequencies named. The process of evaluation of M_r and of the polarization is described. It is confirmed that a critical collision frequency ν_c equal to $p_1^2 - 2p_1$ gives a transition from propagation with the characteristics of that transverse to the magnetic field (occurring when $0 \leqslant \nu < \nu_c$) in which the polarization is linear when the electron-density $N = 3\pi mf^2/2e^2$ (the polarization of each of the basic modes changing sign on passing through linearity), to that with the characteristics of longitudinal propagation (occurring when $\nu > \nu_c$) in which the polarization is never linear and retains its sign for all values of N in both modes.

The attenuation and absorption are found to be, in general, greater for the right-handed component than for the left-handed component, with the direction of magnetic field appropriate for down-coming waves in the northern hemisphere. A table is given showing the polarization of the component having the greater index of attenuation and the greater value of $\kappa\mu$ with varying electronic density. The use of the dispersion curves in the interpretation of propagation phenomena is discussed; consideration of the values of the indices of attenuation in addition to those of the indices of refraction leads to the general conclusion that the lower boundary of the Kennelly-Heaviside layer must be sharp in the optical sense and that reflection of long waves occurs at this boundary at the same height, in general, as that of much shorter waves.

§ 1. INTRODUCTION

The first part of this paper* was concerned with a discussion of the Appleton-Hartree formula giving the indices of refraction and attenuation for propagation of electromagnetic waves in an ionized medium with an external magnetic field. In this formula the symbol $M = \mu - i\kappa c/p$; μ is the index of

M, μ

* *Proc. Phys. Soc.* 45, 245–64 (1933); subsequently referred to as "part 1."

refraction; $i = \sqrt{(-1)}$; κ the index of attenuation; c is the velocity of electromagnetic waves in vacuo; $p = 2\pi \times$ the frequency of the waves under consideration; $\alpha = -p^2/{p_0}^2 - \frac{1}{3}$; $\beta = p\nu/{p_0}^2$; ${p_0}^2 = 4\pi Ne^2/m$; $\nu = 1/\tau$ and is the frequency of collision of electrons with molecules; N is the electron-density; e, m are the charge and mass of an electron; $\gamma_T = pp_T/{p_0}^2$; $\gamma_L = pp_L/{p_0}^2$; $p_T = H_T e/mc$; $p_L = H_L e/mc$; and H_T, H_L are the components of magnetic field perpendicular to and along the direction of propagation. With these symbols the formula was as follows

i, κ, c p $\alpha, \beta, p_0, \nu,$ N, e, m γ_T, γ_L, p_1 p_L H_T, H_L

$$M^2 = \left(\mu - \frac{i\kappa c}{p}\right)^2 = 1 + \frac{2}{2(\alpha + i\beta) - \dfrac{{\gamma_T}^2}{1 + \alpha + i\beta} \pm \left(\dfrac{{\gamma_T}^4}{(1 + \alpha + i\beta)^2} + 4{\gamma_L}^2\right)^{\frac{1}{2}}};$$

the discussion of part 1 was restricted to the phenomena holding in the absence of collisional friction between the free electrons, which are assumed to be the cause of the dispersion, and the molecules of the surrounding gas. This implied a zero value of the collision frequency ν and therefore of the quantity β; the expression for M^2 was purely real and the modes of propagation which could occur were (i) and (iii) described in part 1, § 2, pp. 248–50. When collisional friction is present ν and β are no longer zero and the expression for M^2 becomes complex; neither μ nor κ can be zero (with the exception that $\kappa = 0$ when $N = 0$) and the mode of propagation is the second one described in part 1, § 2, in which progressive waves are propagated but suffer exponential diminution of amplitude, which falls to $1/e$ of its value in a distance κ^{-1}. This type of propagation is associated with true absorption of energy, the amount being proportional to the product $\mu\kappa$*. For values of ν of the order of 10^5 per sec. or less, the values of μ and κ for certain ranges of values of the electron-density approach very nearly to zero, and the presence of friction merely has the effect of removing the infinities in the dispersion curves and rounding off the corners slightly; so that the dispersion curves given in part 1 may, as was mentioned in the discussion, be used in the interpretation of propagation phenomena when the value of ν is estimated to be of the order of 10^5 per sec. or less. As ν increases, the steep maxima and minima of the dispersion curves are rapidly smoothed out and the curves for different wave-frequencies become more similar to one another; at the critical collision frequency ν_c or ${p_T}^2/2p_L$ the propagation changes from the type described in part 1, § 4, p. 263 (in which the polarization of each basic mode is linear when $N = \frac{3}{2}\pi mf^2/e^2$ and changes there from left-handed to right-handed or *vice versa*) to a type approximating to that of transmission along the external magnetic field, in which the polarization is never linear and that of either basic mode retains its sense of rotation for all values of N. This confirms the conclusion of E. V. Appleton and G. Builder†. For still greater values of ν, the curves for μ and κ as functions of N for any given frequency tend more and more to linearity and approach the values 1 and 0 respectively; these are their limits for an infinite degree of friction, in which case the medium exerts no refractive or dispersive effect upon the waves passing through it, since the electrons are then unable to execute forced oscillations.

ν_c

* Cf. part 1, p. 249.

† *Proc. Phys. Soc.* **45**, 217–18 (1933).

The dispersion curves given in this paper illustrate the effect of various degrees of friction at different wave frequencies. They have been drawn from calculations* of M_r^2 as a function of N, the electron density, with the value of the imposed magnetic field used in part 1 and values of ν, the collision frequency, increasing from zero, where M_r^2 is $(\mu_r - i\kappa_r c/p)^2$. — M_r

Four typical wave-frequencies have been chosen, one from the range of each of the four classes of dispersion curve described in part 1; the corresponding wavelengths are 80, 240, 400 and 1000 m., for which the dispersion curves for $\nu = 0$ were given in part 1. The values of ν which have been found to be representative of different orders of magnitude of the effect of friction are 10^5, 10^6, and 10^7 c. sec. Friction exerts practically no influence on propagation, even of the longer waves, until the collision frequency is of the order of the frequencies used in practical wireless communication (about 100 kc. sec.). When ν has reached a value of 10^7 c./sec., μ and κ are practically linear functions of N for any one frequency, and the type of propagation is almost exactly longitudinal. $\nu = 10^6$ c./sec. gives an intermediate value.

§ 2. EVALUATION OF M_r AND CRITICAL COLLISION FREQUENCY

In calculating the values of M_r, the first quantity to be evaluated is

$$p_0^4 [\gamma_T^4 + 4\gamma_L^2 (1 + \alpha + i\beta)^2]^{\frac{1}{2}} = [p^4 p_T^4 + 4p^2 p_L^2 (\tfrac{2}{3} p_0^2 - p^2 + ip\nu)^2]^{\frac{1}{2}};$$

it is from the alternative signs to be prefixed to this square root that the two values M_a and M_b are obtained. In part 1 the quantity under the square root sign was always positive, being the sum of the squares of two real quantities; the values M_a and M_b corresponding to the ordinary and extraordinary waves were given by the positive and negative signs respectively. With β and ν unequal to 0, the quantity under the square-root sign is complex and the vector OP representing it moves away from the real axis into the Argand diagram, as shown in figure 1. — M_a, M_b

The square root of the complex quantity represented by OP, or $re^{i\theta}$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, is represented by the two vectors OP' or $r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ and OP'' or $r^{\frac{1}{2}} e^{i\frac{1}{2}(\theta + 2\pi)}$.

As OP tends to the x-axis ($\theta \to 0$), OP' tends to lie along the positive direction of Ox and OP'' along its negative direction, so that, for continuity with the case of zero friction, we must, for small values of β and N, take OP' as the root corresponding to M_a and OP'' as that corresponding to M_b; i.e. the roots giving M_a and M_b are those whose real parts are positive and negative respectively.

The curve described by P in the Argand diagram is given by

$$\begin{cases} x = p^4 (p_T^4 - 4p_L^2 \nu^2) + 4p^2 p_L^2 (\tfrac{2}{3} p_0^2 - p^2)^2, \\ y = 8p^3 p_L^2 \nu (\tfrac{2}{3} p_0^2 - p^2). \end{cases}$$

* The calculations have included the term which, according to Hartree, expresses the discreteness of the scattering centres, so that the equation $\chi = p^2 p_0^2 - \frac{1}{3}$ has been used as against the equation $\chi = p^2 p_0^2$ which holds in the absence of this term. The inclusion of the term is still a debatable matter; if it be omitted, $N = \pi m f^2/e^2$ should replace $N = \frac{3}{2} \pi m f^2/e^2$ throughout.

The parameter N, the variable electronic density, occurs only in the quantity $p_0^2 = 4\pi Ne^2/m$; eliminating $\frac{2}{3}p_0^2 - p^2$, we find that the locus of P is the parabola

$$y^2 = 16p^4 p_L^2 \nu^2 [x - p^4 (p_T^4 - 4p_L^2 \nu^2)]$$

with axis along Ox.

For N and therefore $p_0 = 0$, $y = -8p^5 p_L^2 \nu$, a negative quantity, so that the parabola begins at a point P below the axis of x. Its vertex A is at the point $[p^4 (p_T^4 - 4p_L^2 \nu^2), 0]$ where $p_0^2 = \frac{3}{2}p^2$ or $1 + \alpha = 0$.

As $N \to \infty$, both x and y become positive and increase without limit. If $\nu < \nu_c$, the x-coordinate of the vertex is positive, figure 2·1; if $\nu = \nu_c$, it is zero, figure 2·2, and if $\nu > \nu_c$ it is negative, figure 2·3. The vectorial square root OP_a of OP corresponding to M_a must, for continuity with the case $\nu = 0$ for small values of N, be such that its real part is positive in every case when $0 > \theta_P > -\pi$ where θ_P is xOP; θ_P in figure 2·1, θ_P steadily increases to zero and positive values and the real part of OP_a remains positive for all N; P_a moves along the curve $P_a A_a Q_a$. In figure 2·2, $\nu = \nu_c$; A, A_a and A_b have moved to coincidence at O and the path of P_a is $P_a O Q_a$. The real part of OP_a is still always positive. Figure 2·3 illustrates the case where $\nu > \nu_c$; OA_a now lies along the negative y-axis and, for continuity, the curve $P_a A_a$ must continue with increasing N into the left half-plane to points such as Q_a. Thus when $\nu > \nu_c$ and $1 + \alpha > 0$, the square root of OP corresponding to M_a must be taken with a negative real part. The paths $P_b A_b Q_b$ of the end of the vector giving the square root corresponding to M_b are shown as dashed lines in the figures.

Considerations of continuity have thus made clear the rule which must be adopted for determining which of the signs prefixed to the real part of the square root gives M_a^2 and which M_b^2; the positive sign must be chosen for M_a^2 except when $\nu > \nu_c$ and $1 + \alpha > 0$, when the negative sign must be taken. By definition, μ_r and κ_r $(r = a, b)$ are always positive, so that the remainder of the calculation is r straightforward. The curves for μ_r and κ_r are shown in figure 3. Those for the real part of M_r^2 are shown in figure 4 and those for the imaginary part of M_r^2 in figure 5.

The critical collision frequency ν_c, i.e. $p_T^2/2p_L$ or $(H_T^2/2H_L)(e/mc)$, at which the dispersion curves change so fundamentally, depends on the values of e, m and c and further only on the value of the external magnetic field and its inclination to the direction of wave-propagation. It is remarkable that it is independent of the wave-frequency and of the electronic density. For the value of the earth's magnetic field at Slough (for which the curves given here have been calculated) and vertical propagation, the value of ν_c is found to be $6{\cdot}86 \times 10^5$ which would correspond to a wave-length of 437·3 m.

When $\nu = \nu_c$, the parts of the dispersion curves beyond

$$N = \tfrac{3}{2}\pi m f^2/e^2 \text{ (i.e. } 1 + \alpha = 0)$$

appear to change over, so that points where μ_a was very nearly zero when $\nu < \nu_c$ are, when $\nu > \nu_c$, on the curve for μ_b and *vice versa*. This is most clearly seen in the curves of figure 4, which show the real part of M_r^2. This is the form in which the dispersion curves given in part 1 were drawn; with $\nu = 0$, M_r^2 was purely real and μ_r and κ_r could be immediately deduced by taking the square root of M_r^2. The

curves for $\nu = 10^5$ c./sec., figures 4·11, 4·21, 4·31, 4·41, are evidently the curves of part 1 with the infinite branches joined up so that the curves become continuous. $\nu = 10^6$ c./sec., figures 4·12, 4·22, 4·32, 4·42, is already above $\nu = \nu_r$ and the branches above $N = 3\pi mf^2/2e^2$ ($1 - \alpha = 0$) have crossed over though there are still kinks in the curves near $N = 3\pi mf^2/2e^2$. For $\nu = 10^7$ c. sec. the effect of friction predominates and the kinks have almost disappeared; it is already clear that the curves tend to $M_r^2 = 1$ as $\nu \to \infty$.

§ 3. POLARIZATION

The formula giving the polarization was stated in part 1, § 2 in the form

E

$$-\frac{E_x}{E_y} = \frac{H_y}{H_x} = \frac{i\gamma_L(1 - M^2)}{1 + (\alpha + i\beta)(1 - M^2)},$$

where the incident wave is propagated along Oz and the external magnetic field

x, y, z has components H_L, H_I along Oz and Ox respectively. Hence, when M_r^2 has been evaluated, the polarization may be deduced. When collisional friction is present, the ratio $-E_x/E_y$ or H_y/H_x is no longer evaluated as a purely imaginary quantity, but contains a real part also; the principal axes of the polarization ellipses no longer

ψ lie along Ox and Oy but are rotated through an angle ψ. It is still true that the product $(H_y/H_x)_{M_a}(H_y/H_x)_{M_b} = 1$, so that the major axes of the ellipses are again mutually perpendicular and the ellipticities are the same, while the directions of description are opposite. Figure 6 illustrates the crossed polarization ellipses

ψ, θ corresponding to M_a and M_b and shows how the angles ψ and θ are defined geometrically. As in part 1, $\theta = 0$ corresponds to linear polarization; positive values of θ correspond to an anti-clockwise (left-hand) direction of rotation viewed along the direction of propagation, and negative values of θ to clockwise (right-hand) rotation. $\theta = \pm 45°$ correspond to left- and right-hand circular polarization respectively.

A, B Let $iE_y/E_x = A + iB$, and be calculated from the formula above. The ellipse described by the end of the electric vector in the (x, y) plane is then given by

$$x = E_x = \cos \omega t,$$

$$y = E_y = R\left(\frac{A + iB}{i} e^{i\omega t}\right) = B \cos \omega t + A \sin \omega t$$

or, eliminating t, $y^2 - 2Bxy + x^2(A^2 + B^2) = A^2$.

The major axis of this ellipse is inclined to Ox at an angle ψ given by

$$\tan 2\psi = 2B/(1 - A^2 - B^2)$$

θ and the ratio of the axes, which we will call $\tan\theta$ as in part 1, is given by

$$\tan^2\theta = \frac{(1 + A^2 + B^2)\sin 2\psi - 2B}{(1 + A^2 + B^2)\sin 2\psi + 2B}.$$

Consider the case in which $A^2 + B^2 \ll 1$, a relation holding for most of the calculations of the polarization corresponding to M_a^2.

The signs of ψ and θ are then determined by the signs of B and A respectively; if $B > 0$, the part of E_y which is in phase with E_x is positive, so that ψ is a positive acute angle; if $B < 0$, ψ is a negative acute angle. For small values of N, ψ_a is found to be small and negative for all wave-frequencies; if $N > 3\pi mf^2/2e^2$ and $\nu < \nu_c$, $\psi \to 0$ again, as shown in figures 6.*s*1 ($s = 1$–4); when $\nu > \nu_c$, ψ_a must for continuity be taken as tending to $-\pi/2$ as shown in figures 6·*st* ($s = 1$–4, $t = 1$–3).

The sign of θ is determined by that of A, which gives the part of E_y out of phase with E_x; if $A < 0$, $\theta < 0$ and rotation from E_x to E_y is clockwise or in a right-handed direction, whereas if $A > 0$, $\theta > 0$ and the rotation is anti-clockwise or left-handed (looking along the positive direction of propagation of the waves). For small values of N, the ordinary and extraordinary waves are found to be left- and right-handedly polarized respectively, as they were for the case of zero friction discussed in part 1. If $\nu < \nu_c$, the numerical value of θ starts at 45° when $N = 0$ and decreases until when $N = 3\pi mf^2/2e^2\,(1 + \alpha = 0)$, $\theta = 0$ and the polarization is linear, corresponding to a real value of the ratio H_y/H_x. When $N > 3\pi mf^2/2e^2$, θ increases numerically again to 45°; the polarization of the ordinary wave corresponding to M_a is now right-handed and that of the extraordinary wave, corresponding to M_b, is left-handed. $|\psi|$ decreases again from its maximum value when $N = 3\pi mf^2/2e^2$ to zero when $N = \infty$, where the waves are circularly polarized. Thus when $\nu < \nu_c$ the polarization phenomena are essentially the same as when $\nu = 0$; the effect of friction is merely to rotate the polarization ellipses through the angle ψ, which is small except when ν is very nearly equal to ν_c. The fact that the polarization passes through the linear form when $N = 3\pi mf^2/2e^2$ shows that the type of propagation is similar to that transverse to the magnetic field when the polarization, viewed along the direction of propagation, is always linear.

In the case when $\nu > \nu_c$, i.e. $2p_L\nu > p_T{}^2$, the ratio H_y/H_x can never be purely real and the polarization never passes through the linear form. If the values of M_a and M_b when $N > 3\pi mf^2/2e^2$ are found as described in § 2, θ_a does not become zero when $N = 3\pi mf^2/2e^2$ but still has a finite positive value and, for greater values of N, remains positive, increasing again towards a limiting value of 45° as $N \to \infty$. Similarly, θ_b remains negative, so that the propagation is now of the type of that longitudinal to the magnetic field, in which the two basic modes have always circular polarization, one consistently left-handed and the other consistently right-handed for all values of N. This is the limit to which the propagation at any angle to the magnetic field tends as $\nu \to \infty$. For the value of the earth's magnetic field for which the curves here given have been calculated, this limit has been very nearly reached when $\nu = 10^7$ c./sec. and even when $\nu = 10^6$ c./sec. there is only a dip in the θ curve near the point where $N = 3\pi mf^2/2e^2$. At the critical collision frequency itself this dip would reach right down to the point where $\theta = 0$ when $N = 3\pi mf^2/2e^2$.

In the curves in figures 6·11 to 6·43 only the values of ψ and θ corresponding to M_a have been drawn as functions of N for the various wave and collision frequencies chosen. It is clear from figure 6 that the major axis of the ellipse corresponding to M_b is rotated through an angle of ψ_a from the y-axis, while $\theta_b = -\theta_a$.

The variations of θ_a and ψ_a described above are clearly shown in the curves, from which it is also evident that the general course of ψ_a and θ_a is the same for all wave-frequencies, as in part 1, and that the critical collision frequency $\nu = \nu_c$ is the decisive factor in determining the type of propagation.

Figure 7 shows the polarization when $N = 3\pi mf^2/2e^2$ $(1 + \alpha = 0)$ as a function of the collision frequency. When $\nu < \nu_c$, $\theta_a = 0$ while ψ_a varies from 0 to $-45°$ at $\nu = \nu_c$. When $\nu > \nu_c$, ψ_a remains constant and equals $-45°$ while θ_a increases from 0, tending to a limit of $45°$ as $\nu \to \infty$.

§ 4. THE DISPERSION CURVES AND THEIR USE IN THE INTERPRETATION OF PROPAGATION PHENOMENA

The dispersion curves given in figure 3 show the calculated values of the indices of refraction, μ_r, and the indices of attenuation, κ_r $(r = a, b)$, for the wave and collision frequencies selected as illustrative of the phenomena. The curves of figure 3·*s*0 $(s = 1\text{–}4)$ show the values of μ_r and κ_r for $\nu = 0$. This is an alternative presentation of the curves shown in part 1, figure 3·1, where the real quantity M_r^2 was shown as a function of N; μ_r and κ_r are real positive quantities and, when $\nu = 0$, are zero over certain ranges of N. The occurrence of total reflection when $\nu = 0$ at the points where $\mu = 0$ was discussed in part 1, §§ 2, 6.

The curves of figures 3·*st* $(s = 1\text{–}4, t = 1\text{–}3)$ show the values of μ_r and κ_r when $\nu = 10^5, 10^6, 10^7$ c./sec. The zero and infinite values of μ_r and κ_r disappear when $\nu \neq 0$ (with the exception that $\kappa_r = 0$ when $N = 0$), and the only mode of propagation which can occur is (ii) described in part 1, § 2, in which the progressive waves suffer exponential diminution of amplitude. The curves of figure 3·*s*1, when $\nu = 10^5$ c./sec., differ however very little from those of figure 3·*s*0 when $\nu = 0$; μ_r very nearly $= 0$ over the ranges in which it was actually zero when $\nu = 0$, and for these values κ_r is large, so that the propagation phenomena when ν is of the order of 10^5 c./sec. will not differ appreciably from those described in part 1 for cases in which $\nu = 0$. When, however, ν has increased to values above ν_c, as in figures 3·*s*2, 3·*s*3, the propagation has changed from the transverse to the longitudinal type and the course of the dispersion curves is completely different. For frequencies above that at which $p = p_H$ (i.e. $\lambda = 227{\cdot}1$ m. for the values taken in the calculations), the first value N_{b_1}*, is no longer an approximate zero of μ_b, which now has values very nearly zero at values of N near N_a; μ_a is very nearly zero at values of N near N_{b_2}, so that the ordinary and extraordinary waves may be expected to require larger values of N to reflect them, if the μ_r curves alone are taken into account; the extraordinary wave would still, however, be reflected before the ordinary one. In the case when friction is present, however, it is necessary to take into consideration also the fact that non-zero values of κ_r occur for all values of μ_r, and, if κ_r is sufficiently large, the amplitude of the wave may have decreased so much before conditions of appreciable reflection are reached that the reflected energy, if any, would be too small to be detected. The amplitude falls to $1/e$ of its

* Cf. part 1, p. 261.

value in a distance κ_r^{-1}; expressing this in terms of the quantity $\kappa_r c/p$, which is directly calculated from M_r^2 and is shown in the curves, we can estimate the attenuation from the relation

$$\kappa_r c/p = \log_e x/2\pi n,$$

where n is the number of wave-lengths *in vacuo* in which the amplitude decreases to $1/x$. Let us consider as a practical example the detection of plane 80-m. waves, reflected vertically from the ionosphere, with a receiver which will show the presence of waves whose amplitude is 0·01 of the amplitude of the waves at their incidence on the ionosphere. Remembering that they have to pass twice through the thickness of ionized medium below the reflection point*, so that $x = 10$, we find that for all frequencies this thickness may be 10 vacuum wave-lengths of the medium (i.e. $n = 10$) if $\kappa_r c/p$ does not exceed $\log_e 10/2\pi \times 10$ or 0·04 approximately.

With the curves of figure 3·12 when $\lambda = 80$ m., $\nu = 10^6$ c./sec., this value is reached when $N = 1{\cdot}1 \times 10^5$ approximately, which is less than the value $2{\cdot}1 \times 10^5$ of N at which the first minimum of μ_b occurs. The value of $\kappa_b c/p$ for this first minimum of μ_b is 0·415 and the amplitude of the waves is attenuated to 0·1 of its value in a space of $0{\cdot}88\lambda$ or about 70 m. Thus, if the ionization N rose to $1{\cdot}1 \times 10^5$ in a distance of 800 m. at a height at which the collision frequency was of the order of 10^6 c./sec., the right-handed (extraordinary) reflected component would be so much attenuated that it would not be detectable at the receiver. If the ionization increases more gradually than this, the appropriate value of $\kappa_r c/p$ will be correspondingly smaller and will be reached for very much smaller values of N. Similar arguments apply to the ordinary (left-handed) component; at the value $2{\cdot}3 \times 10^5$ of N which corresponds to attenuation to 0·01 of the emitted amplitude for a layer-thickness of 800 m., μ_a is still 0·48. The approximate reflection point for the ordinary component when $\nu = 10^6$ c./sec. is that at which $N = 4 \times 10^5$, corresponding to a value 0·19 of $\kappa_a c/p$ and complete absorption would occur in a layer of this density of thickness $1{\cdot}93\lambda$ or 154 m. approximately. The general conclusion is that if either component of waves of length 80 m. is to be reflected from the height where $\nu = 10^6$ c./sec., the increase in ionization to the values required for reflection must occur within a distance of the order of a wave-length, so that the reflection approximates to that studied in geometrical optics. This value of ν occurs† at a height of 90 km. approximately. Thus the lower boundary of the Kennelly-Heaviside layer must be sharp and there must be no absorbing layer of appreciable thickness below it. For if such a layer occurred at a height of say 70 km., where ν is estimated‡ to be of the order of 10^7 c./sec., the curves of figure 3.13 show that the right-handed component would be absorbed if this layer were 800 m. thick and of density $N = 2 \times 10^4$. If reflection takes place at greater heights, say 110 km., where § ν is thought to be of the order of 10^5 c./sec., the curves of figure 3·11

* The waves are not strictly reflected from a point, but from a region in which μ approximates to zero; this will however only alter the result by a factor of the order 2, which does not affect the order of magnitude of the conclusion.

† S. Chapman, *Proc. R.S.* A **122**, 373 (1929).

‡ *Loc. cit.*

§ *Loc. cit.*

give the reflection conditions in this case. Here $\kappa_r c/p$ remains very small up to the points where μ_r is very nearly zero, so that the ionization could increase more gradually to its reflection value for either component without the occurrence of appreciable absorption. It would, however, still be necessary that no absorbing layers should occur at lower levels.

A similar study may be made of the curves showing the effect of friction at lower frequencies; figures 3·21 and 3·22, for which $\lambda = 240$ m., show that the right-handed component will be completely absorbed before N approximates at all to the values required to reflect it, whereas reflection of the left-handed component may be expected from the point where $N = 3 \times 10^4$ when ν is of the order of 10^5 c./sec., or where $N = 6 \times 10^4$ (with much smaller intensity) when $\nu = 10^6$ c./sec.

Figures 3·31 and 3·32, for which $\lambda = 400$ m., show the same increased absorption of the extraordinary wave, which, even for $\nu = 10^5$ c. sec., would be absorbed before its reflection point where $N = 2{\cdot}9 \times 10^4$ was reached, owing to the rise in $\kappa_b c/p$ for values of N of the order of 10^4. When $\nu = 10^6$ the extraordinary wave would be very quickly absorbed, while if the ordinary wave succeeded in reaching the value 3×10^4 of N required to reflect it (which would be possible if the ionization gradient were sufficiently high) it would be very much attenuated. Figure 3·33 shows that reflection of 400-m. waves from ionized layers at the heights at which $\nu = 10^7$ could only occur if the layer boundary were very sharp.

The dispersion curves for which $\lambda = 1000$ m. are shown in figure 3·4; those for which $\nu = 10^3$ c./sec. and 10^4 c. sec. have also been calculated but differ very slightly from those for which $\nu = 0$. The curves in this range of frequency ($\lambda > 615{\cdot}5$ m.; class 4 of part 1) differ from those for which $\lambda < 615{\cdot}5$ m. in that the left-handed component is the one which is more attenuated for values of N up to $N = 3\pi mf^2/2e^2$ ($1 + \alpha = 0$) and small values of ν. When $\nu = 10^5$ c./sec., as in the case of the curve here shown, the attenuation of the right-handed component is beginning to increase and when $\nu = 10^7$ c. sec. it is the more attenuated for all values of N, so that the behaviour for values of ν greater than ν_c is similar to that of waves of higher frequency. Figure 3·41 shows that the left-handed component of waves of length 1000 m. could, when $\nu = 10^5$, be reflected where $N = 0{\cdot}167 \times 10^4$ ($1 + \alpha = 0$) and the right-handed one from where $N = 10^4$ without serious absorption; figure 3·42 however shows that the left-handed component requires when $\nu = 10^6$ a reflection density $1{\cdot}5 \times 10^4$ of N, and has a strong absorption band round the point where $N = 0{\cdot}167 \times 10^4$, while the right-handed component for which μ_b approximates to zero when $N = 4 \times 10^4$ is absorbed before N has reached this value. When $\nu = 10^7$ c./sec. both components are rapidly absorbed. It is therefore probable that reflection of waves of length 1000 m. occurs at the lower boundary of the Kennelly-Heaviside layer at the same height in general as that of much shorter waves, and not (except perhaps in very exceptional cases) at a height of about 50 km. as has been suggested, where ν would be greater than 10^7 and absorption very strong.

R, L The table shows the component, right-handed (R) or left-handed (L), which

Table 1. Polarizations of components having the greater values of kc/p and $k\mu$

λ (metres)	$\nu = 10^3$				$\nu = 10^4$				$\nu = 10^5$				$\nu = 10^6$				$\nu = 10^7$				$\nu = 10^8$			
	Range of N	Polarization (kc/p greater)	Range of N	Polarization ($k\mu$ greater)	Range of N	Polarization (kc/p greater)	Range of N	Polarization ($k\mu$ greater)	Range of N	Polarization (kc/p greater)	Range of N	Polarization ($k\mu$ greater)	Range of N	Polarization (kc/p greater)	Range of N	Polarization ($k\mu$ greater)	Range of N	Polarization (kc/p greater)	Range of N	Polarization ($k\mu$ greater)	Range of N	Polarization (kc/p greater)	Range of N	Polarization ($k\mu$ greater)
30: typical of wave-lengths from 0 to 227·1	—	—	—	—	—	—	—	—	Up to $2{\cdot}62\times10^5$ ($1+\alpha=0$)	R	Up to $2{\cdot}62\times10^5$ ($1+\alpha=0$)	R	All	R	All	R	All	R	All	R	All	R*	All	R*
									$2{\cdot}62\times10^5$ to $2{\cdot}64\times10^5$	L	$2{\cdot}62\times10^5$ to $2{\cdot}71\times10^5$	L												
									Above $2{\cdot}64\times10^5$	R	Above $2{\cdot}71\times10^5$	R												
240: typical of wave-lengths from 227·1 to 256·2	—	—	—	—	—	—	—	—	Up to $2{\cdot}84\times10^4$	R	Up to $2{\cdot}9\times10^4$ ($1+\alpha=0$)	R	All	R	All	R	All	R	All	R	—	—	—	—
									$2{\cdot}84\times10^4$ to $2{\cdot}9\times10^4$ ($1+\alpha=0$)	L	$2{\cdot}9\times10^4$ to $2{\cdot}98\times10^4$	L												
									Above $2{\cdot}9\times10^4$	R	Above $2{\cdot}98\times10^4$	R												
400: typical of wave-lengths from 256·2 to 615·5	—	—	—	—	Up to $0{\cdot}96\times10^4$	R	All values except about $1{\cdot}03\times10^4$ to $1{\cdot}046\times10^4$	R	Up to $0{\cdot}96\times10^4$	R	All	R	All	R	All	R	All	R	All	R	—	—	—	—
					$0{\cdot}96\times10^4$ to $1{\cdot}046\times10^4$ ($1+\alpha=0$)	L			$0{\cdot}96\times10^4$ to $1{\cdot}046\times10^4$ ($1+\alpha=0$)	L																
					Above $1{\cdot}046\times10^4$	R	About $1{\cdot}03\times10^4$ to $1{\cdot}046\times10^4$	L†	Above $1{\cdot}046\times10^4$	R														
1000: typical of wave-lengths from 615·5 to ∞	Up to $0{\cdot}1674\times10^4$ ($1+\alpha=0$)	L	Up to $0{\cdot}1674\times10^4$ ($1+\alpha=0$)	L	Up to $0{\cdot}1674\times10^4$ ($1+\alpha=0$)	L	Up to $0{\cdot}1674\times10^4$ ($1+\alpha=0$)	L	Up to $0{\cdot}08\times10^4$	L	Up to $0{\cdot}1674\times10^4$ ($1+\alpha=0$)	L	Up to $0{\cdot}1\times10^4$	†	All	R	All	R	All	R	—	—	—	—
	Above $0{\cdot}1674\times10^4$	‡R	Above $0{\cdot}1674\times10^4$	‡R	Above $0{\cdot}1674\times10^4$	‡R	Above $0{\cdot}1674\times10^4$	‡R	$0{\cdot}08\times10^4$ to $0{\cdot}14\times10^4$	R	Above $0{\cdot}1674\times10^4$ ($1+\alpha=0$)	‡R	$0{\cdot}1\times10^4$ to $0{\cdot}1674\times10^4$	L										
									$0{\cdot}14\times10^4$ to $0{\cdot}1674\times10^4$ ($1+\alpha=0$)	L			$1{\cdot}67\times10^3$ to $1{\cdot}42\times10^4$	R										
									Above $0{\cdot}1674\times10^4$	‡R			Above $1{\cdot}42\times10^4$	L										

* Though the two are very nearly equal.

† Components nearly equal.

‡ Owing to change-over of polarization of ordinary wave.

corresponds to the greater index of attenuation and also to the greater value of the quantity $\kappa_r \mu_r$, which is proportional to the amount of energy actually absorbed. In general, the right-handed component is the more attenuated and absorbed except for certain small ranges of N and for waves of length greater than 615·5 m. at very small collision frequencies. Since these long waves are reflected or absorbed before they reach the high levels at which ν is of the order of 10^4 or less, this higher attenuation of the left-handed component is of little importance in the interpretation of practical results.

The curves of figure 5, showing the values of the quantities $\mu_r \kappa_r$, display absorption maxima in certain regions of N. For frequencies belonging to classes 1 and 3 of part 1, figures 5·1, 5·3, there is only one maximum of $\kappa_a \mu_a$ and one of $\kappa_b \mu_b$ when $\nu = 10^5$, 10^6 per sec., but for those belonging to class 2, part 1, figure 5·2, there are two maxima of $\kappa_b \mu_b$ when $\nu = 10^5$ per sec. and for class 4, part 1, figure 5·4, there are two maxima of $\kappa_a \mu_a$ when $\nu = 10^5$ per sec. When $\nu = 10^7$ per sec. $\kappa_r \mu_r$ is practically a linear function of N. A logarithmic scale has been chosen for these curves, and also for those of figure 3·4 (μ_r and κ_r when $\lambda = 1000$ m.), as the range of values to be included was so great.

§ 5. CONCLUSION

The curves given in the present paper illustrate the two outstanding phenomena of magneto-ionic propagation in the presence of collisional friction:

(1) Attenuation and absorption are, in general, greater for the right-handed than for the left-handed component, with the value of magnetic field appropriate for down-coming waves in the northern hemisphere.

(2) The critical collision frequency ν_c or $p_T^2 2p_L$ gives the transition from propagation with the characteristics of that transverse to the magnetic field (occurring when $0 \leqslant \nu \leqslant \nu_c$) to that with the characteristics of longitudinal propagation (which occurs for $\nu > \nu_c$).

The curves may be used for calculations of reflection coefficient, attenuation and absorption in practical cases for the wave-lengths and collision frequencies for which they have been constructed, and they illustrate the general type of phenomenon which is to be expected at other frequencies and intermediate collision frequencies.

§ 6. ACKNOWLEDGMENTS

This work was carried out at the Radio Research Station, Slough, as part of the programme of the Radio Research Board of the Department of Scientific and Industrial Research, and is published by permission of the Board. Thanks are due to Mr R. A. Watson Watt, Superintendent of the Station, for his interest in the work and for the provision of facilities for conducting it; to Prof. E. V. Appleton and Prof. D. R. Hartree for valuable discussion, and suggestions as to the presentation of the paper; to Dr L. J. Comrie, for advice on the arrangement of the calculations; to Miss A. C. Stickland for assistance in the calculations and to Mr E. C. Slow for help in drawing the curves.

DISCUSSION

Prof. E. V. Appleton. I do not think it is possible to over-estimate the usefulness of the author's elucidation of the magneto-ionic formula to experimental workers in this field. The wave-lengths chosen for the graphical illustration are most suitable and illustrate the variety of phenomena we are to expect.

I gather that the author agrees with the distinction between quasi-longitudinal and quasi-transverse propagation, and agrees that whether the one or the other occurs depends on the magnitude of the critical ratio $2p_L\nu/p_T^2$. Now it is found that, for the case of the penetration of the lower ionized region, propagation is of the quasi-transverse type. This means, therefore, that at a level of about 100 km., the collision frequency of electrons with gas molecules must be less than ν_c (i.e. less than $6{\cdot}86 \times 10^5$ per sec.). Prof. Pedersen's estimate for the collisional frequency at this height is 10^5 per sec.

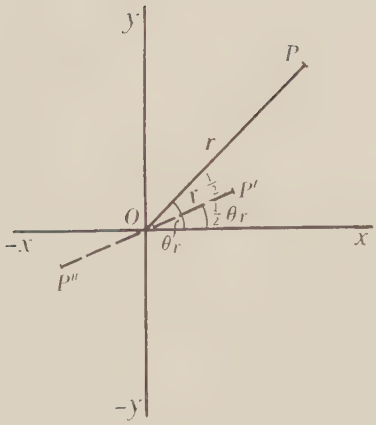


Figure 1.

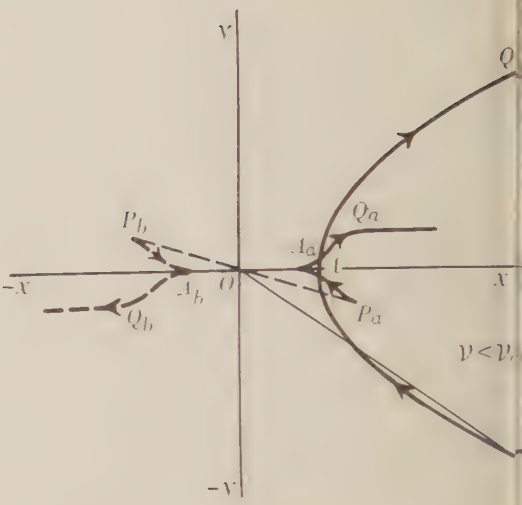


Figure 2·1.

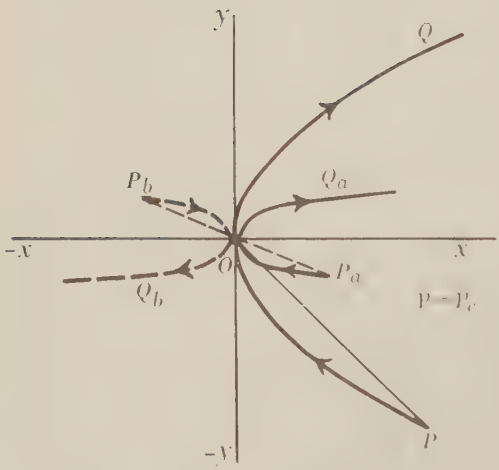


Figure 2·2.

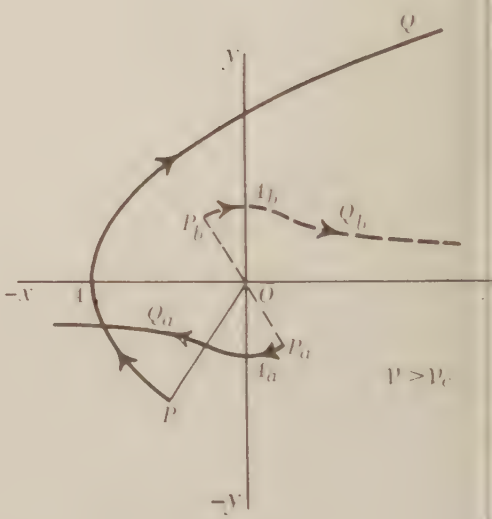


Figure 2·3.

Figure 1. Illustrating the square roots OP', OP'', of the vector OP.

Figure 2. The square roots of OP for: Figure 2·1, $\nu<\nu_c$; Figure 2·2, $\nu=\nu_c$; Figure 2·3, $\nu>\nu_c$ } ——— $r=a$. – – – – $r=b$.

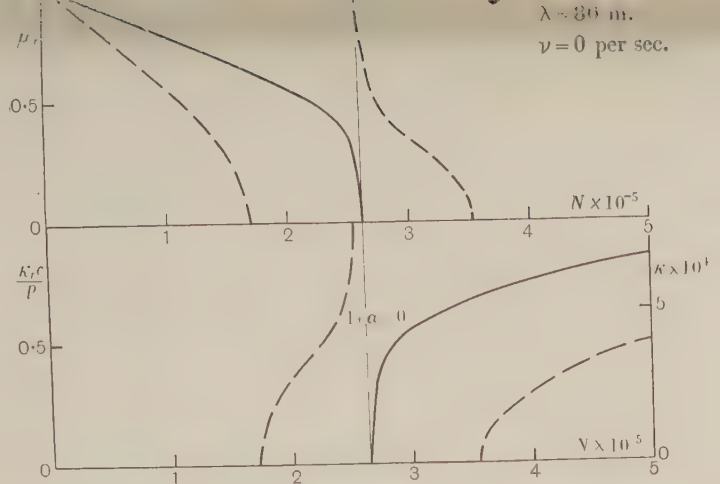


Figure 3·10.

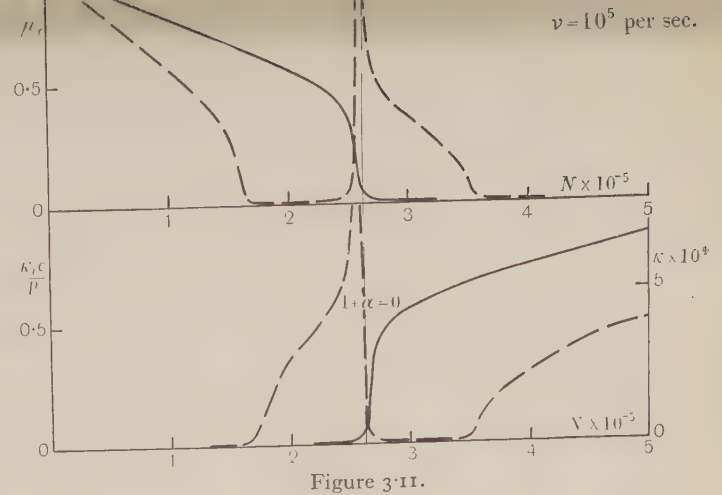


Figure 3·11.

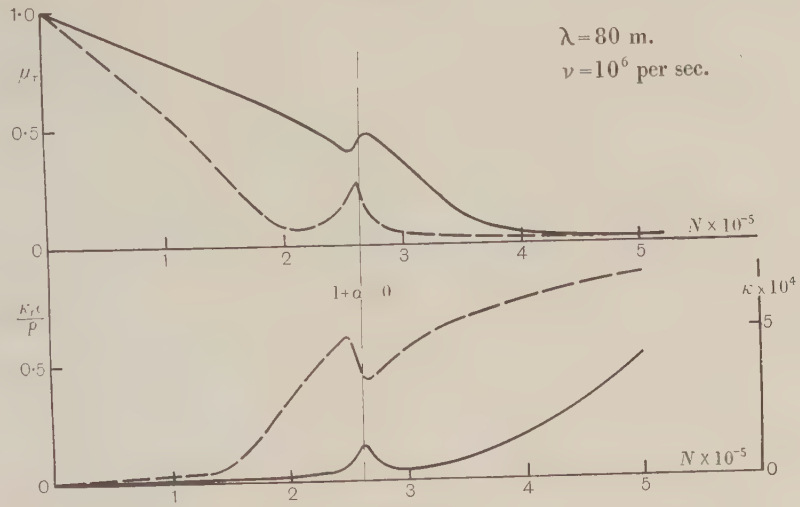


Figure 3·12.

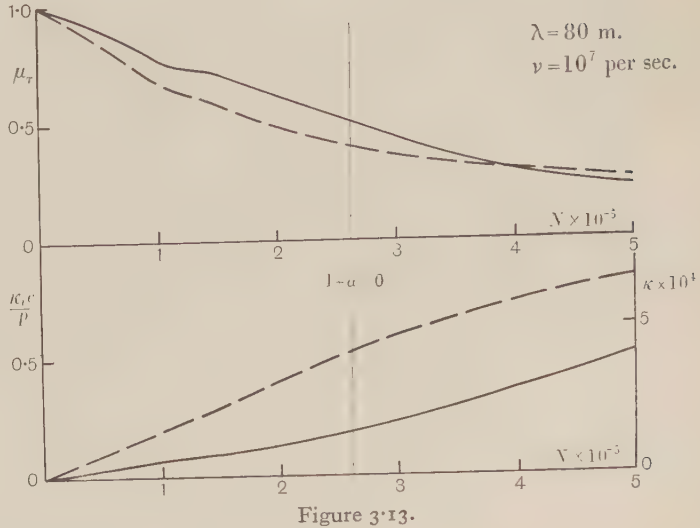
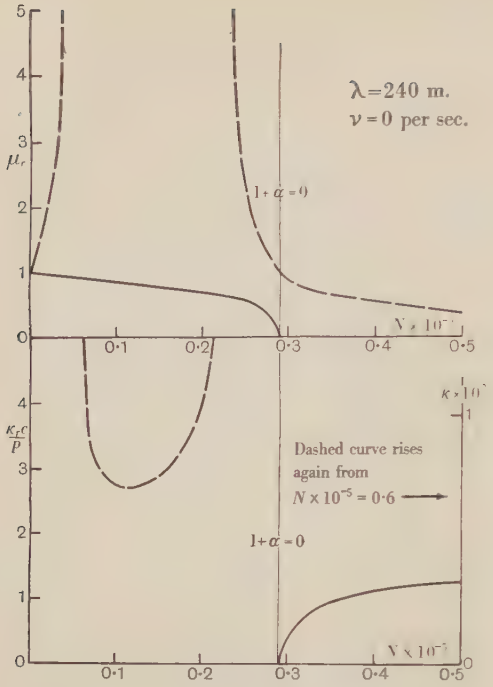


Figure 3·13.

Figure 3·1. Values of μ_r, $\kappa_r c/p$ and κ_r when $\lambda = 80$ m.; $\nu = 0, 10^5, 10^6$ and 10^7 per sec.

$r = a$ ——— $r = b$ — — —



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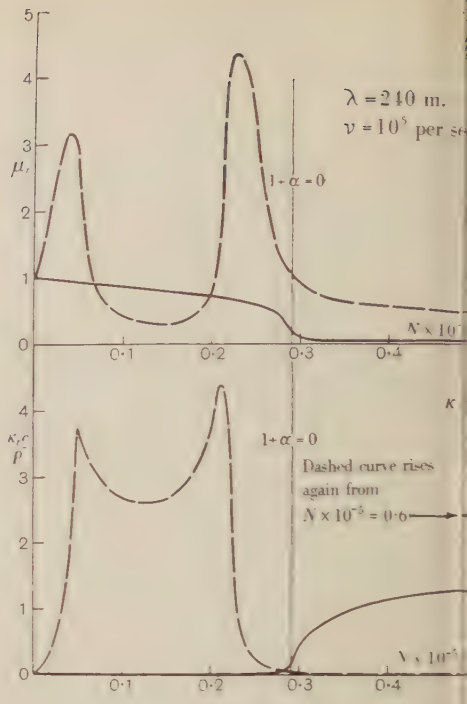

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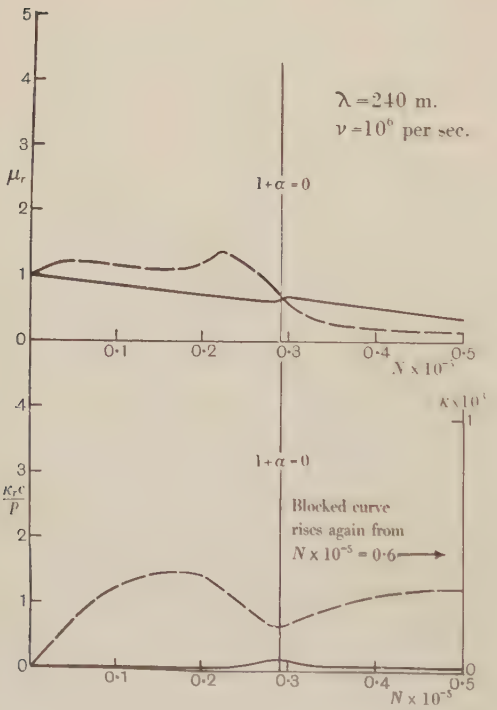

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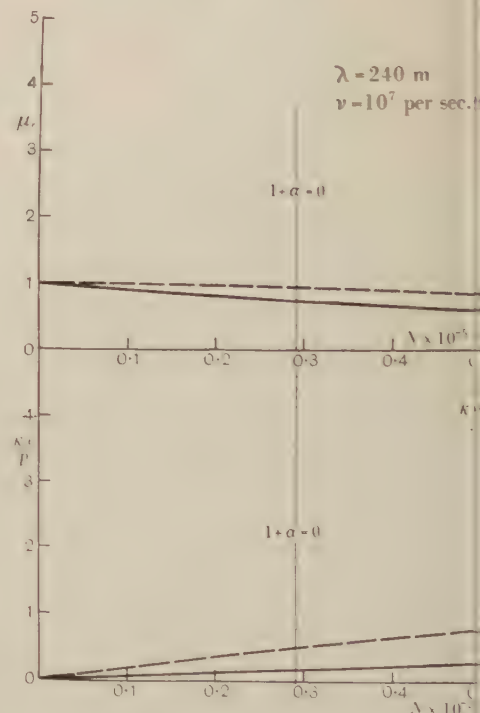


Figure 3·23.

Figure 3·2. Values of μ_r, $\kappa_r c/p$ and κ_r when $\lambda = 240$ m.; $\nu = 0$, 10^5, 10^6, 10^7 per sec.

$r=a$ ——— $r=b$ — — —

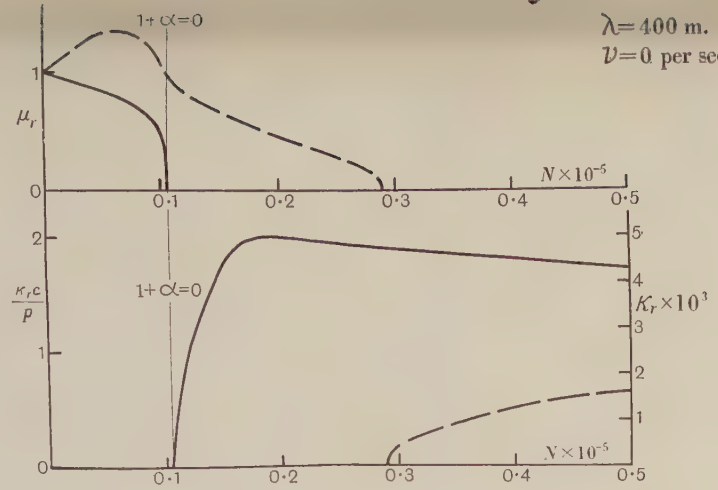


Figure 3·30.

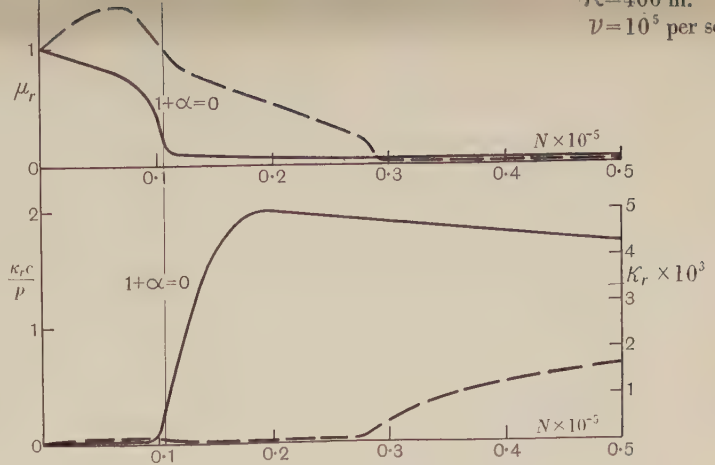


Figure 3·31.

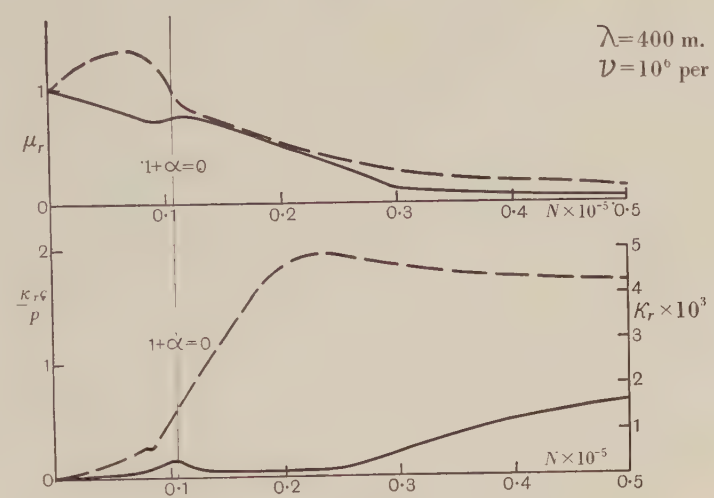


Figure 3·32.

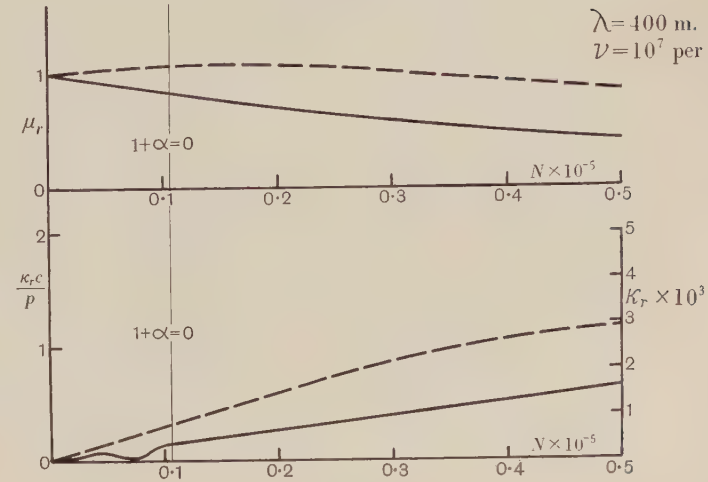


Figure 3·33.

Figure 3·3. Values of μ_r, $\kappa_r c/p$ and κ_r when $\lambda = 400$ m., $\nu = 0$, 10^5, 10^6, 10^7 per sec. $r=a$ ——— $r=b$ – – – –

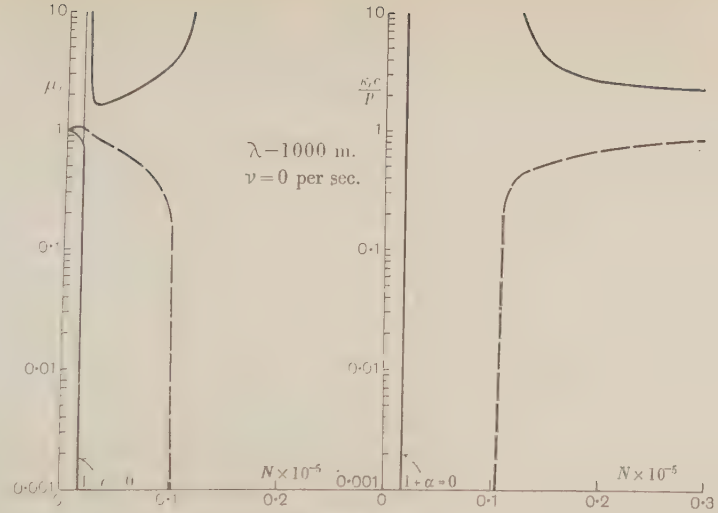


Figure 3·40.

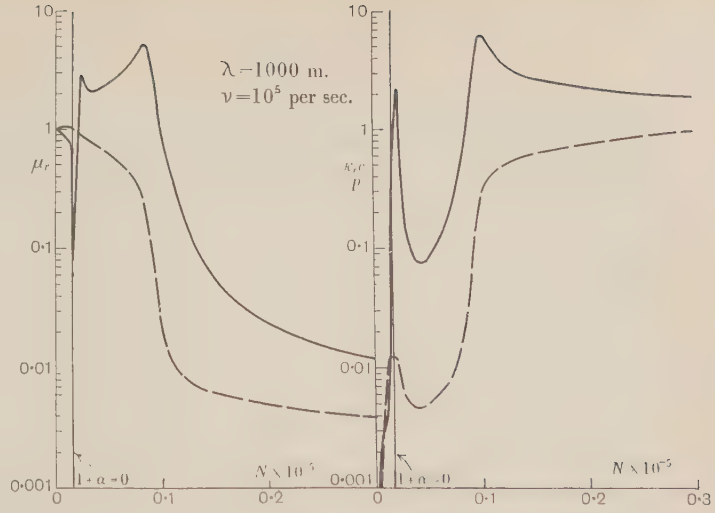
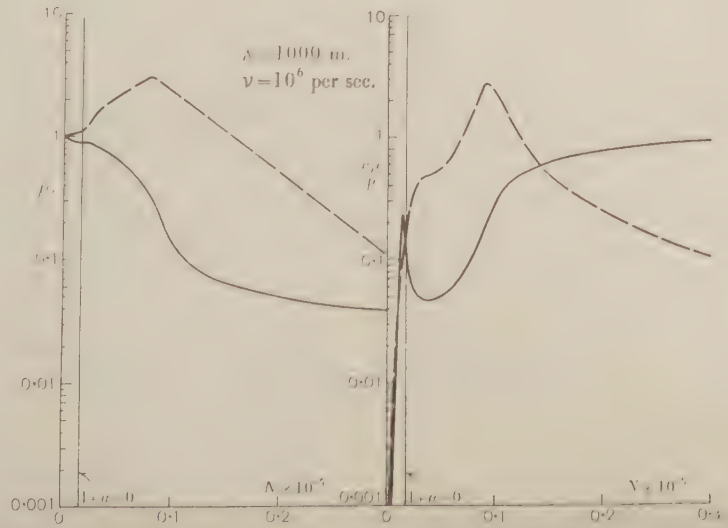
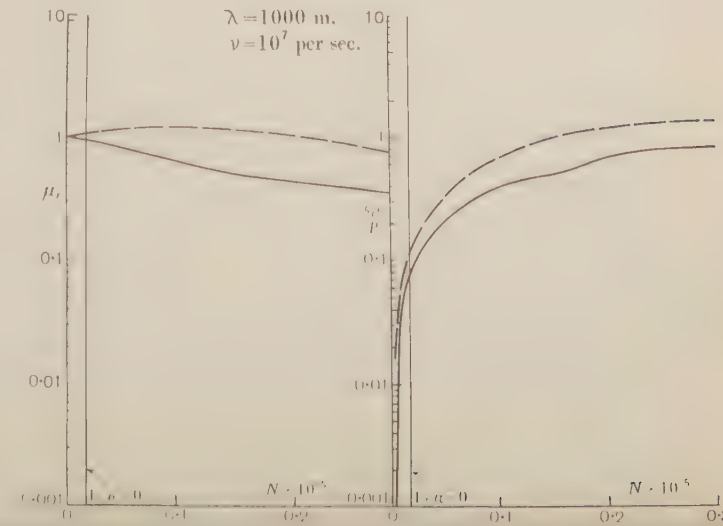


Figure 3·41.





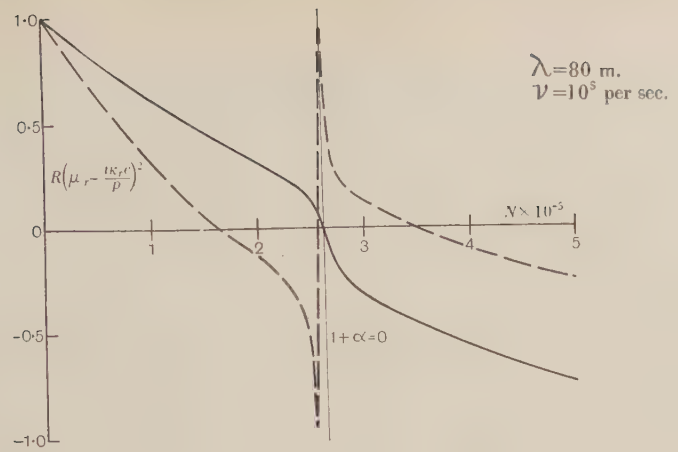


Figure 4·11.

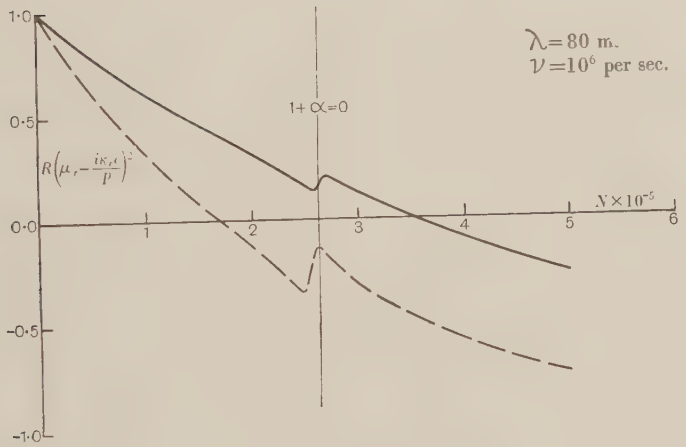


Figure 4·12.

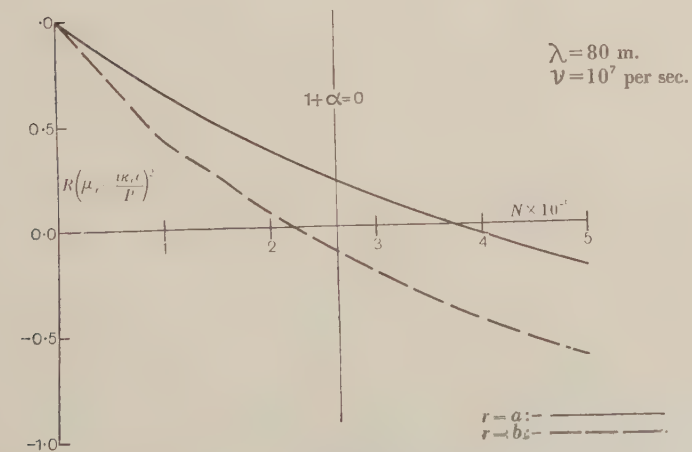
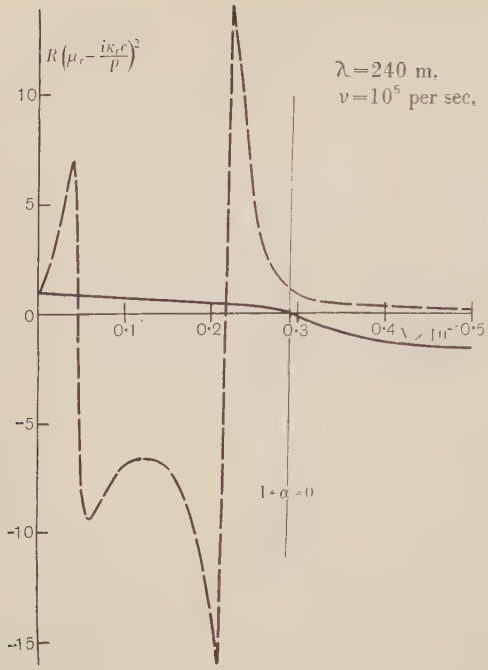


Figure 4·13.

Figure 4·1. Values of $R(\mu_r - i\kappa_r c/p)^2$ when $\lambda = 80$ m.; $\nu = 10^5, 10^6, 10^7$, per sec.



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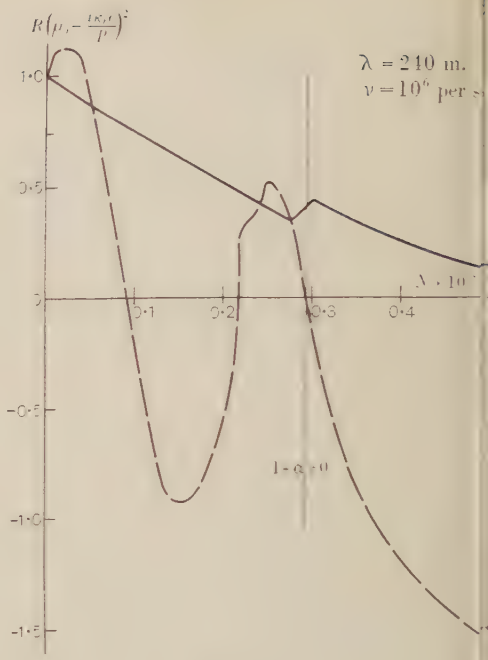

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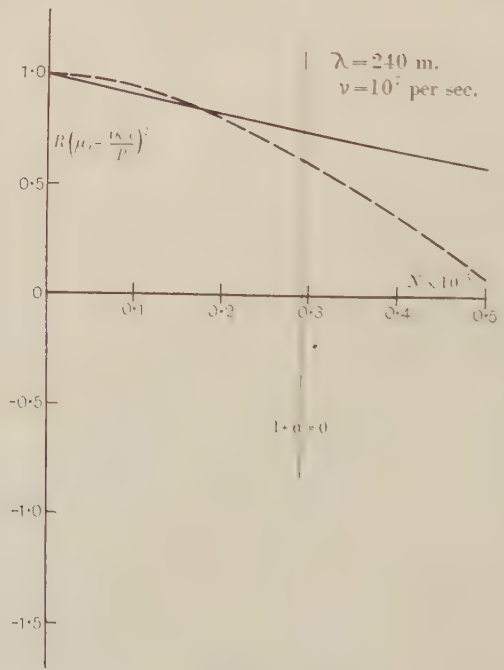


Figure 4·23.

Figure 4·2. Values of $R(\mu_r - i\kappa_r c/p)^2$ when λ=240 m.; ν=10⁵, 10⁶, 10⁷ per sec.

$r=a$ ——— $r=b$ — — — —

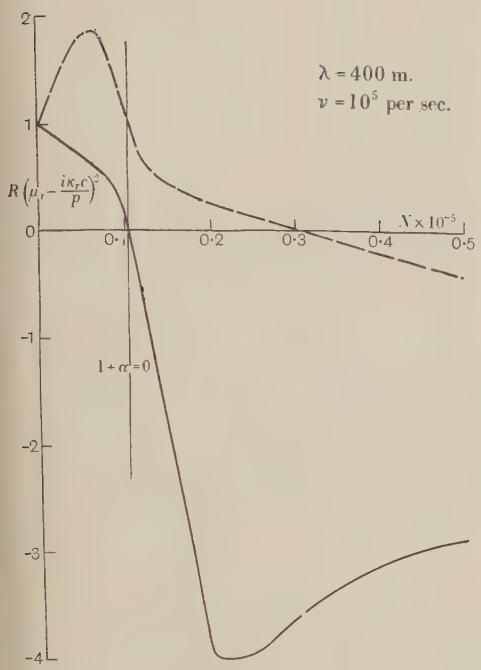


Figure 4·31.

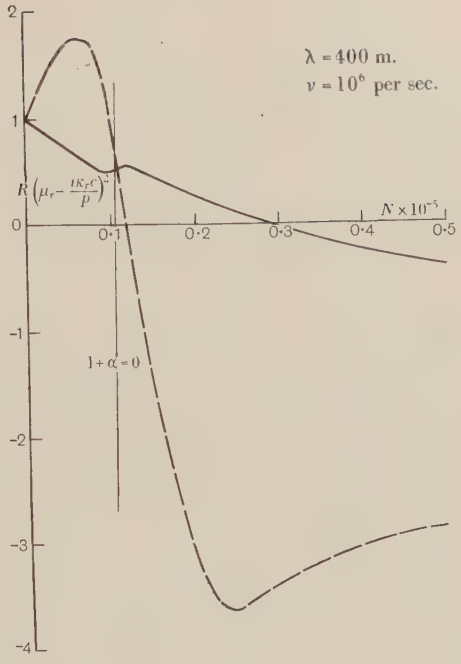


Figure 4·32.

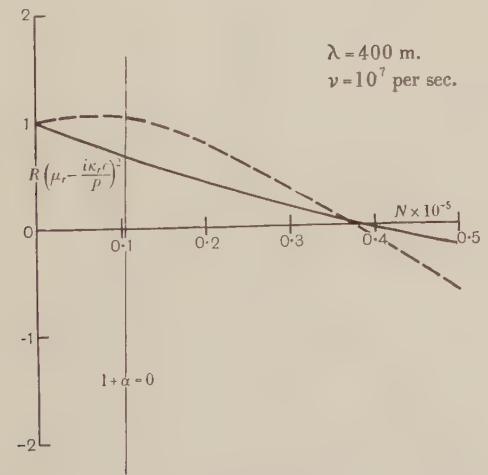


Figure 4·33.

Figure 4·3. Values of $R\,(\mu_r - i\kappa_r c/p)^2$ when $\lambda = 400$ m.; $\nu = 10^5$, 10^6, 10^7 per sec.
$r = a$ ——— $r = b$ — — —

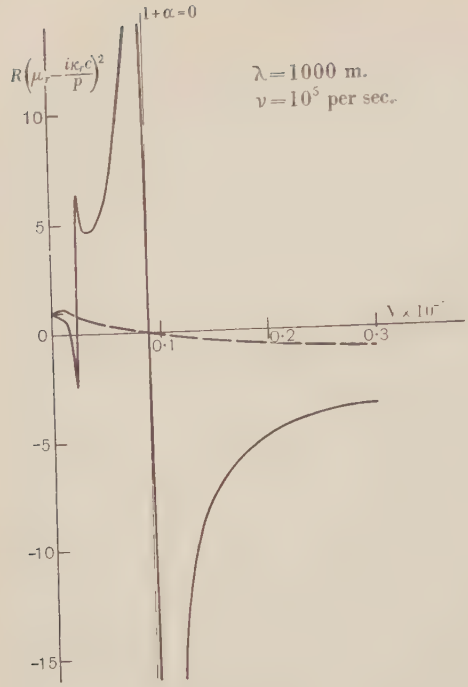


Figure 4·41.

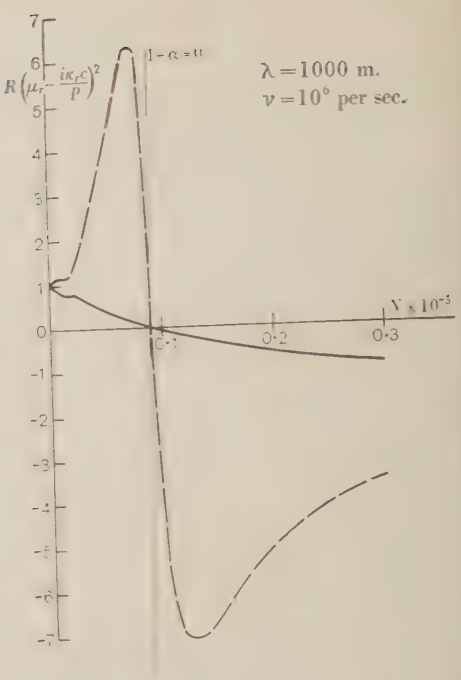
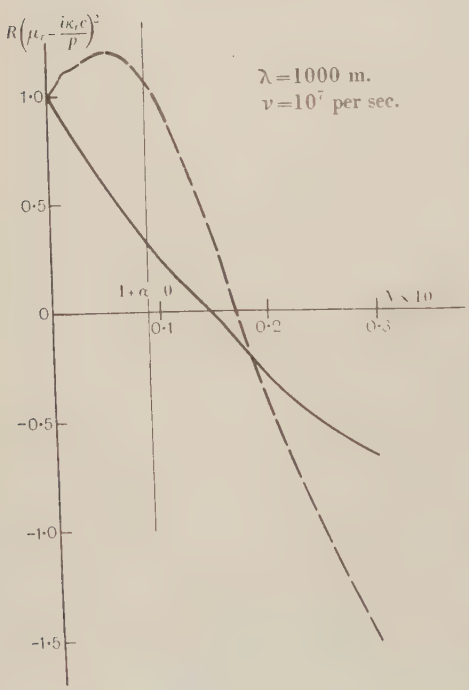


Figure 4·42.



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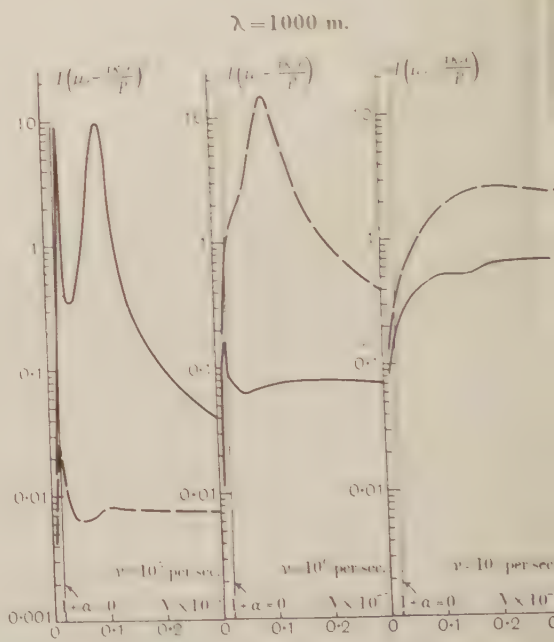


Figure 5·41. Figure 5·42. Figure 5·43.

Figure 4·4. Values of $R(\mu_r - i\kappa_r c/p)^2$ when $\lambda=1000$ m.; $\nu=10^5$, 10^6, 10^7 per sec. $r=a$ ——

Figure 5·4. Value of $I(\mu_r - i\kappa_r c/p)^2$ when $\lambda=1000$ m.; $\nu=10^5$, 10^6, 10^7 per sec. $r=b$ — — —

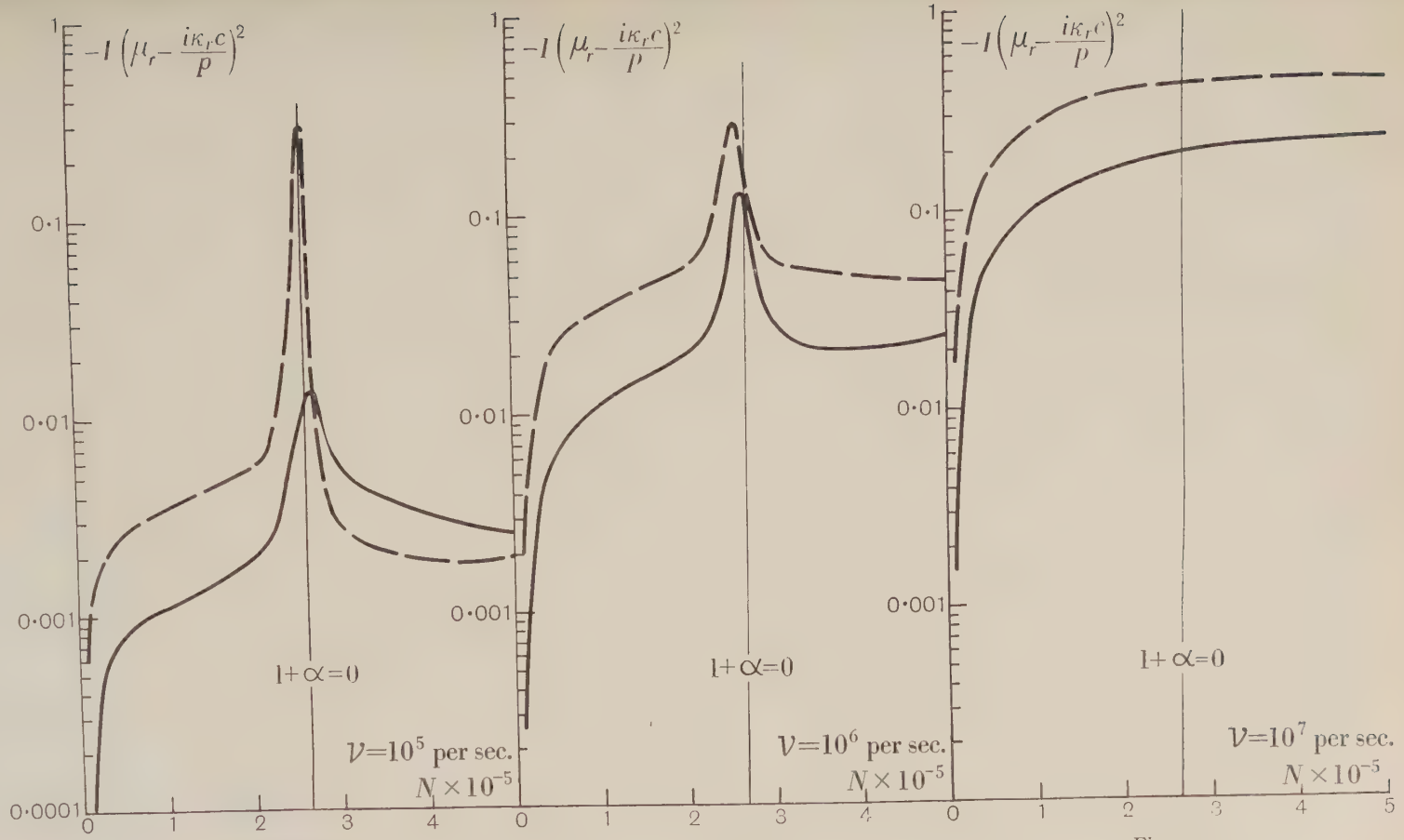
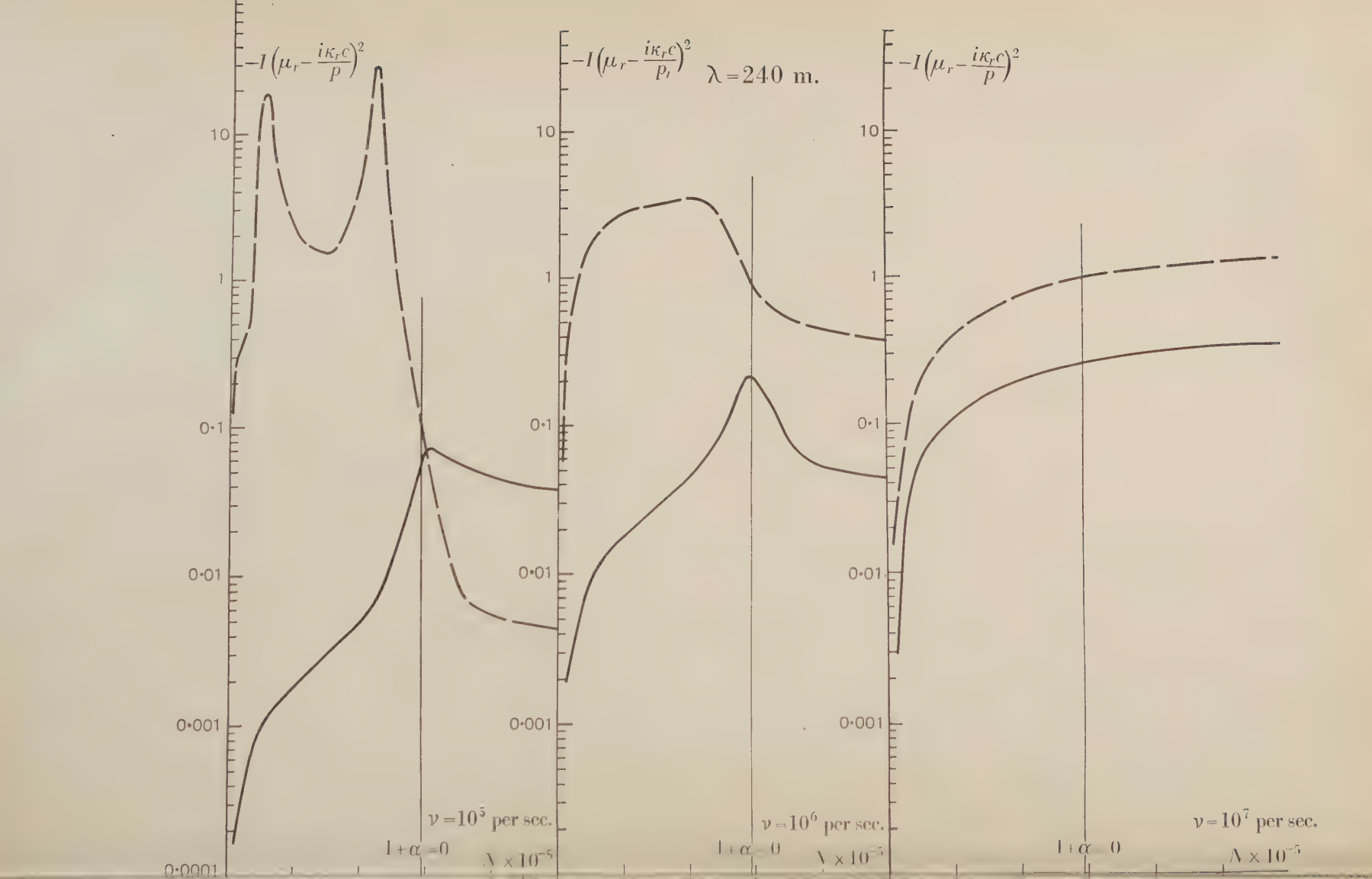


Figure 5·11. Figure 5·12. Figure 5·13.

Figure 5·1. Values of $-I(\mu_r - i\kappa_r c/p)^2$ when $\lambda = 80$ m.; $\nu = 10^5$, 10^6, 10^7 per sec. $r=a$ ——— $r=b$ — — —


$-I\left(\mu_r - \frac{i\kappa_r c}{p}\right)^2$
$-I\left(\mu_r - \frac{i\kappa_r c}{p_l}\right)^2$
$\lambda = 240$ m.
10
1
0·1
0·01
0·001
0·0001
$\nu = 10^5$ per sec.
$\nu = 10^6$ per sec.
$\nu = 10^7$ per sec.
$1 + \alpha = 0$
$\Lambda \times 10^{-5}$

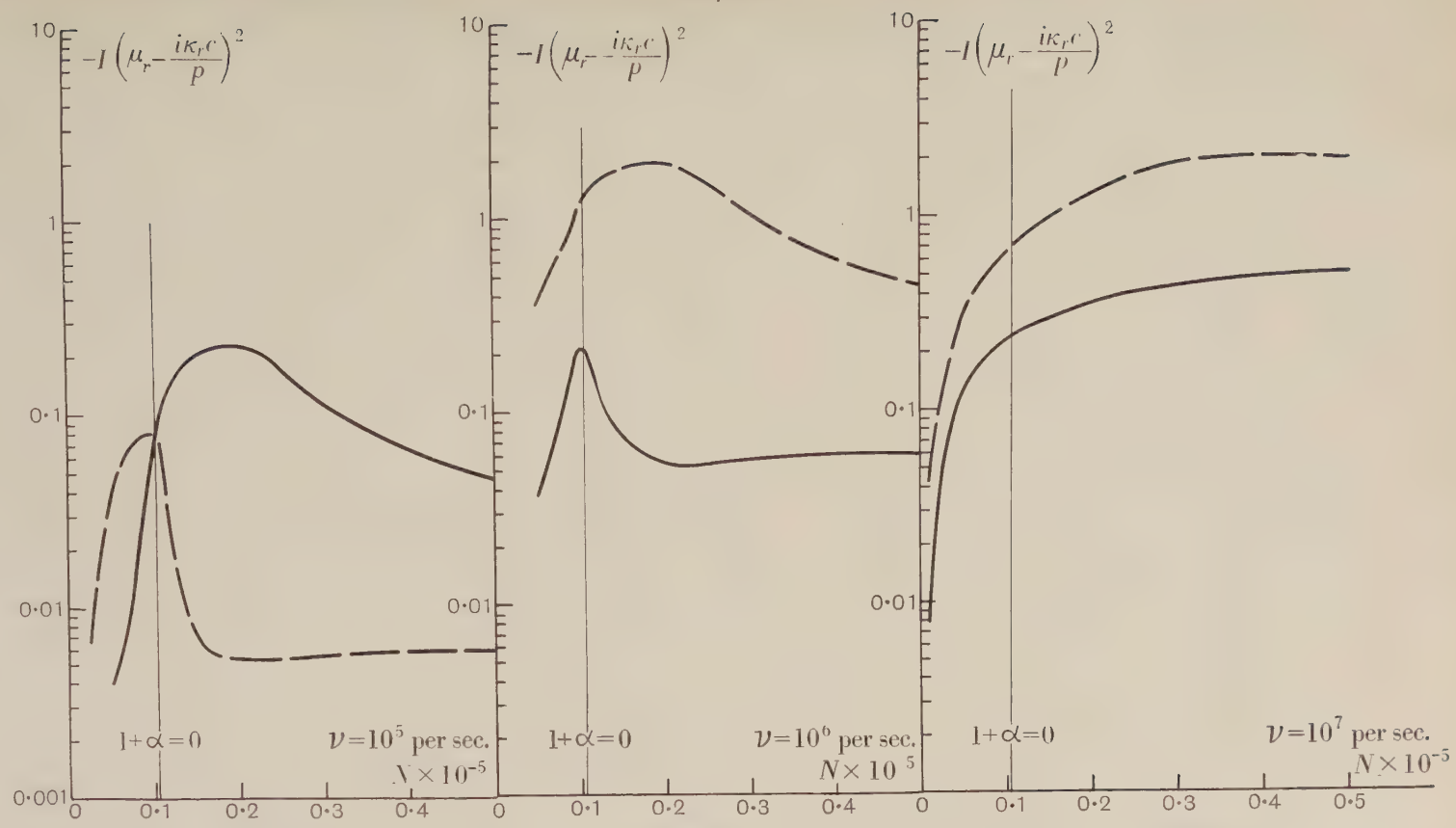


Figure 5·31. Figure 5·32. Figure 5·33.

Figure 5·3. Values of $-I(\mu_r - i\kappa_r c/p)^2$ when $\lambda = 400$ m.; $\nu = 10^5$, 10^6, 10^7 per sec. $r=a$ ——— $r=b$ – – –

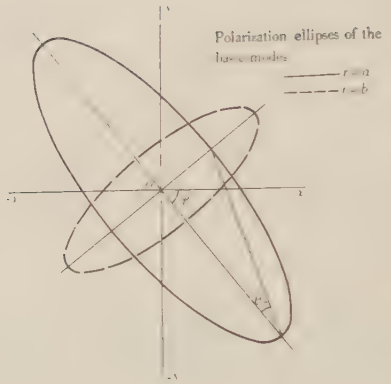


Figure 6.

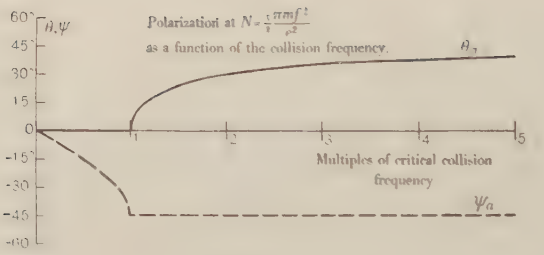


Figure 7.

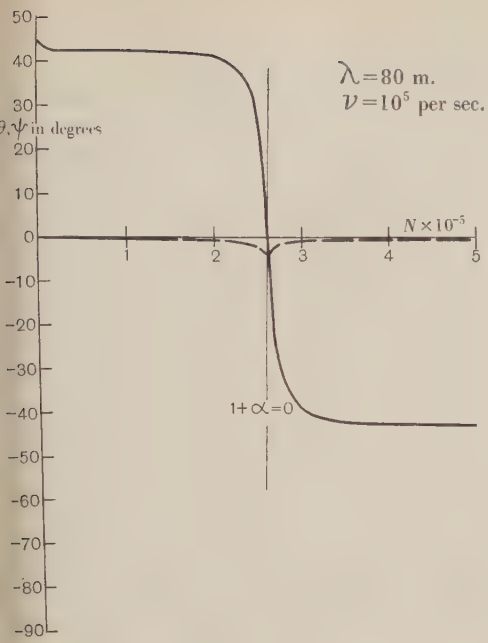
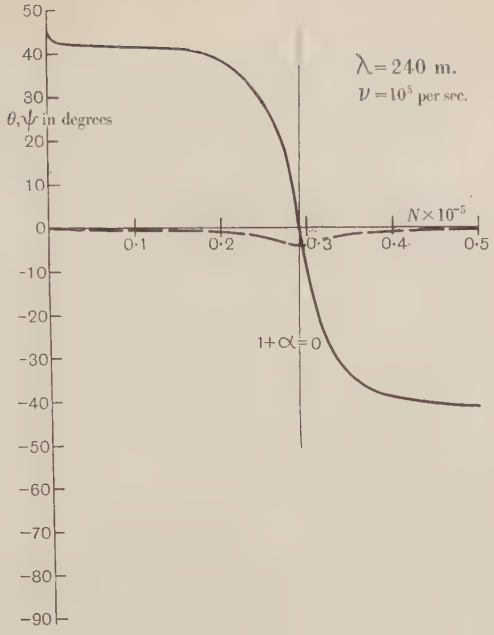


Figure 6·11.



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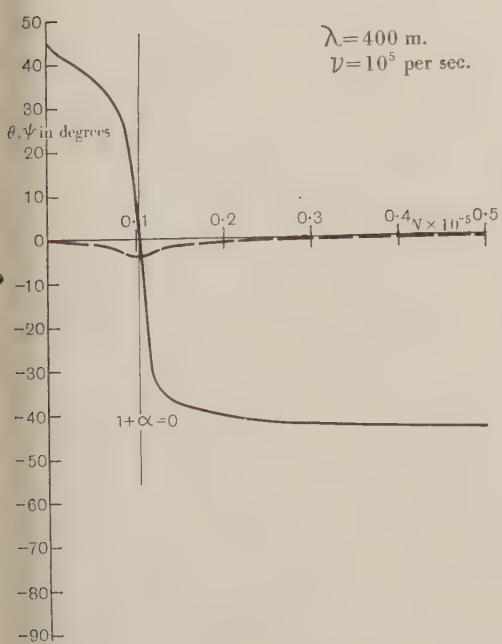


Figure 6·31.

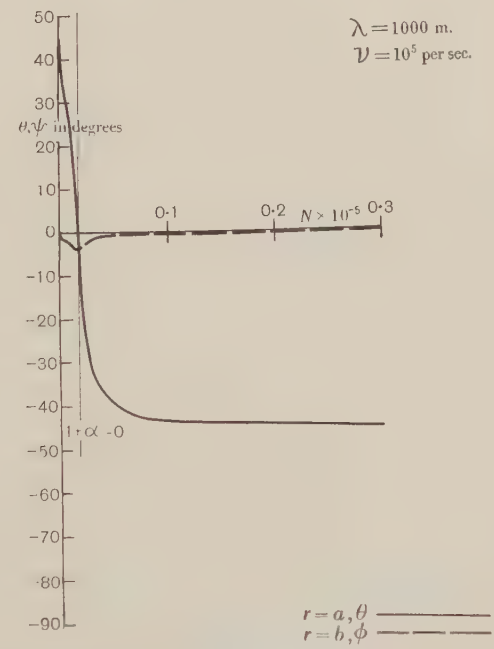


Figure 6·41.

Figure 6·11. Values of θ and ϕ when $\lambda = 80$ m.
6·21. ,, ,, ,, ,, $\lambda = 240$ m.
6·31. ,, ,, ,, ,, $\lambda = 400$ m.
6·41. ,, ,, ,, ,, $\lambda = 1000$ m.
$\nu = 10^5$ per sec.

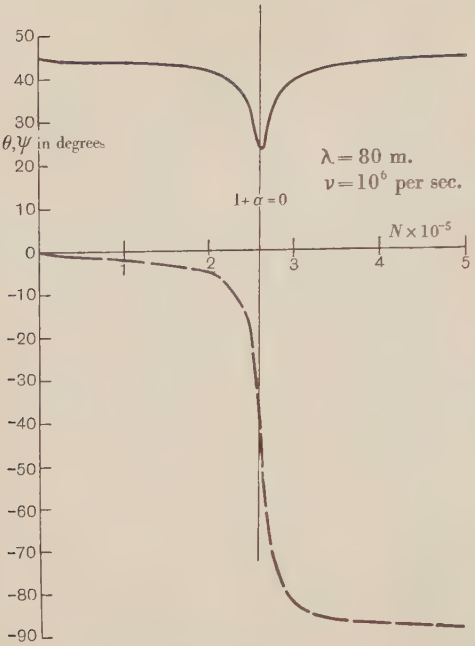
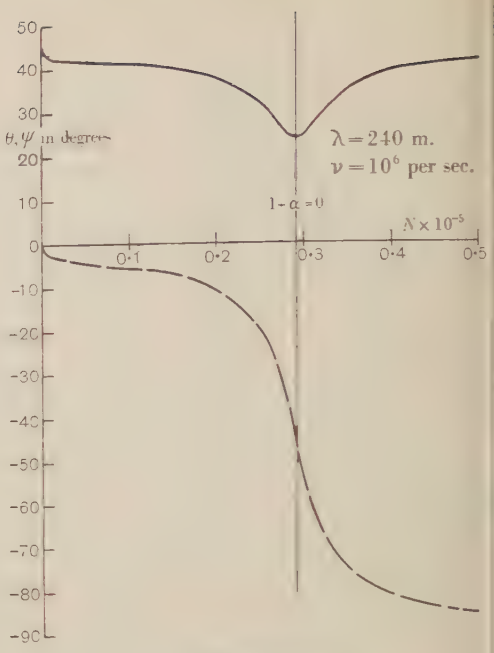


Figure 6·12.



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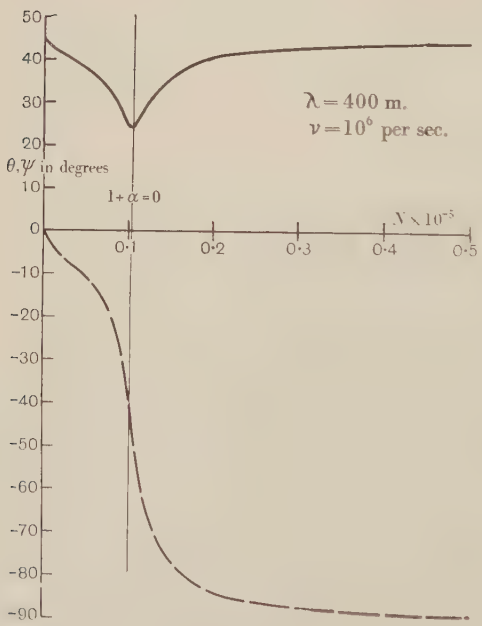

Figure 6·32.

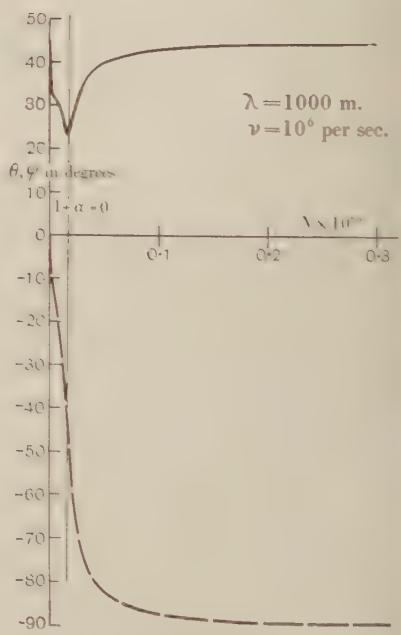

Figure 6·42.

Figure 6·12. Values of θ and ϕ when $\lambda = 80$ m.
6·22. " " " " $\lambda = 240$ m.
6·32. " " " " $\lambda = 400$ m.
6·42. " " " " $\lambda = 1000$ m.
$\nu = 10^6$ per sec.

$r = a$, θ ——— $r = b$, ϕ — — —

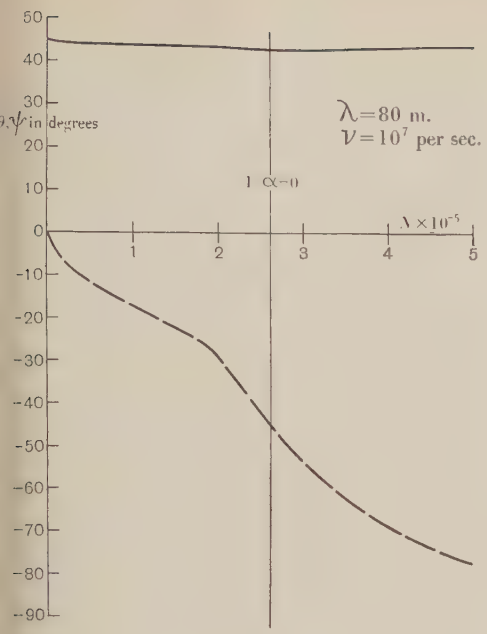

Figure 6·13.

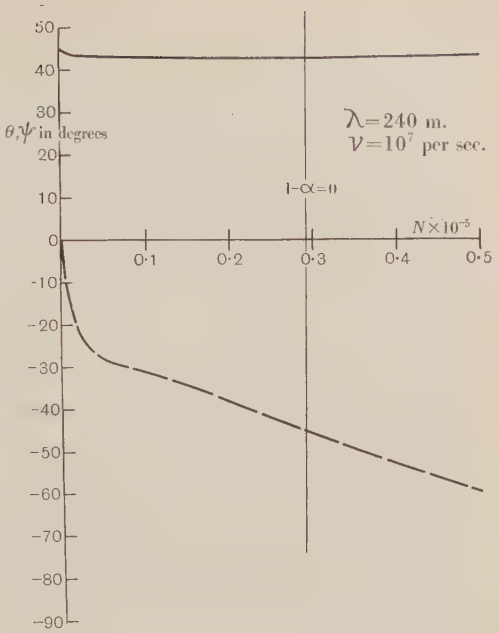

Figure 6·23

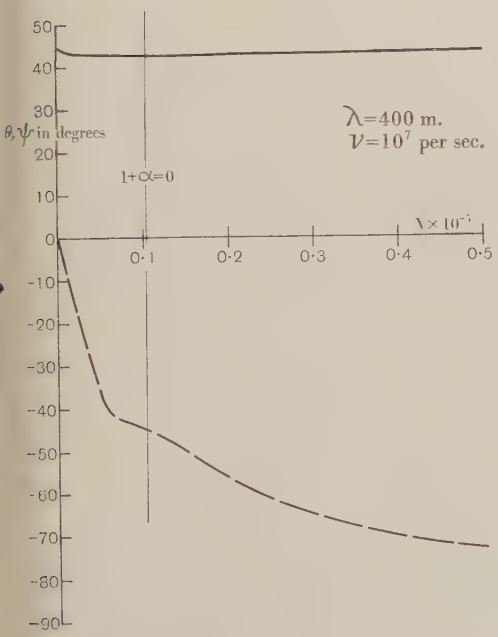

Figure 6·33.

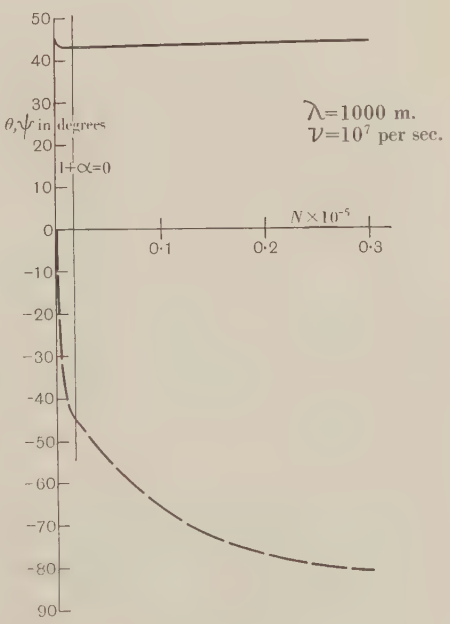

Figure 6·43.

Figure 6·13. Values of θ and ψ when $\lambda = 80$ m.
6·23. ,, ,, ,, ,, $\lambda = 240$ m.
6·33. ,, ,, ,, ,, $\lambda = 400$ m.
6·43. ,, ,, ,, ,, $\lambda = 1000$ m.
} $\nu = 10^7$ per sec.

$r=a$, θ ——— $r=b$, ϕ — — —

535·337:546·17

THE EFFECTIVE ROTATION TEMPERATURE OF THE NEGATIVE GLOW IN NITROGEN

By N. THOMPSON, B.Sc., Physics Department, The University, Sheffield

Communicated by Prof. S. R. Milner, December 12, 1933. Read March 16, 1934.

ABSTRACT. The effective rotation temperature has been determined from the intensity distribution in the (0, 0) band of the nitrogen negative system under varying conditions of excitation. The specially designed discharge tubes used for exciting the spectrum are described and the method of calculating the results explained. These results show that the effective temperature increases slightly with the pressure and current-strength, and to a much greater extent with the temperature of the furnace surrounding the discharge tube. At high temperatures it becomes less than the temperature of the furnace, and an explanation of this surprising behaviour is sought. It is concluded that, in this particular case at least, the effective temperature is not identical with the gas temperature though it depends in part on this quantity; it depends also on other factors not yet accounted for in the theory.

§ 1. INTRODUCTION

It is well known that from the rotational-intensity-distribution in a band spectrum a value for the temperature of the source of the radiation can be deduced even in cases where thermal equilibrium would not be expected, as, for example, in a gas discharge at low pressures. Very diverse values, ranging from 300 to 5000° K. have been found for this effective temperature τ in different gases and with different methods of excitation*. In particular, the so-called nitrogen negative bands have received a considerable amount of attention in this connexion, and even in this one case τ varies over a range from 380 to 1000° K., depending on the conditions of excitation†. There does not appear to have been, however, any systematic enquiry into the effect on τ of these conditions, and in fact the precise significance of that quantity itself is a little vague. The experiments about to be described represent an attempt to throw some light on the second of these questions by investigating the first.

§ 2. EXPERIMENTAL

The measurements were made on the (0, 0) nitrogen negative band at 3914 Å., the spectrum being photographed in the first order of a ten-foot Rowland grating. The density marks for calibrating the plates were obtained by using a step slit on the grating itself; the normal astigmatism of the Rowland mounting was corrected

* J. C. McLennan and others, *Proc. R.S.* **116**, 277 (1927). These results are approximate only, since the quantities given as "intensities" are really photographic densities. W. Lochte-Holtegreven, *Z. f. Phys.* **64**, 443 (1930); **67**, 590 (1931). R. C. Johnson and R. C. Turner, *Proc. R.S.* A **142**, 574 (1933).

† W. R. Van Wijk, *Proc. Amst.* **32**, 1243 (1929); *Z. f. Phys.* **59**, 313 (1930). L. S. Ornstein and W. R. Van Wijk, *Z. f. Phys.* **49**, 315 (1928). A. E. Lindh, *Z. f. Phys.* **67**, 67 (1931).

by means of a convergent cylindrical lens, placed between the slit and the grating, in such a position that the second focal line from the grating, which gives a stigmatic image and which normally lies behind the photographic plate, was brought into focus. The photographs were photometered on a recording microphotometer of the Moll type. A calculation showed that the instrument width of a line in the spectrum was much greater than the natural width, and therefore that peak intensities could be used instead of the theoretically more accurate integrated area intensities. This conclusion was checked by determining both quantities in one or two cases. The peak intensities were found in the usual way, and the areas with the aid of a device already described*. The ratio of the one to the other was found to be constant except for the first three or four lines of the R branch, which were not resolved from lines at the end of the P branch, and hence were not used in the final measurements. Only the lines as far as $R(21)$ could be measured, since $R(22)$ is masked by the head of the (1, 1) band. No results at all were obtained from the P branch, since this forms the head of the band, and the resolution was insufficient for intensity measurements.

Two different types of discharge tube were used for the excitation of the bands. The first is shown diagrammatically in figure 1. The tube A itself was of steel, and into this slipped a lining tube B made of hard glass. This covered the whole of the interior surface of the steel tube with the exception of the conical part C. Into this was pushed a truncated cone of brass gauze, the two together forming the cathode, which was earthed. The anode was a ring of copper wire D in the annular space at the bottom of the tube. In this manner the usual hollow-cathode method of producing the bands was retained, giving a considerable intensity of light. The positive column was confined almost entirely to the annular space, and the second positive group of bands, which in part overlaps the negative group, was only very faint. The presence of the brass gauze was of great assistance in producing this result. The arrangement also ensured that the gas that was emitting the light was close to the walls of the tube, which possessed a high thermal conductivity, while the tube was of such a shape and so supported that it could easily be placed in a constant-temperature bath, or in a furnace, as shown in the figure. The anode lead E was surrounded by a glass insulating sheath which also served to conduct the inflowing gas down to the bottom of the tube. A continuous flow of gas was maintained through the apparatus while a photograph was being taken, the supply being from a cylinder of commercial nitrogen dried by passing over phosphorus pentoxide. A copper-constantan thermocouple was inserted through the bottom of the furnace as at F to determine its temperature, and for some of the observations a second and similar couple was interwoven among the meshes of the gauze forming part of the cathode, as shown.

The second form of tube consisted of a piece of the same steel tubing, mounted horizontally and closed at the ends by glass plates. In this case the tube acted as the anode, and the cathode consisted of a spiral of nickel wire, 1 cm. in diameter

* N. Thompson, *Proc. Phys. Soc.* **45**, 441 (1933); but on p. 443, line 5, the two focal lengths, given in error as 50 and 100 cm., should be 25 and 50 cm. respectively.

and 5 cm. long, coaxial with the anode and in the centre of its ength. There was in addition along the axis of the cathode a fine platinum wire which was used as a resistance thermometer. The furnace wire was wound directly on the central portion of the tube, and the thermocouple was allowed to lie loose inside. In one form, leads were taken from both ends of the cathode as well as from the middle, so that its resistance could be measured while the discharge was passing and its

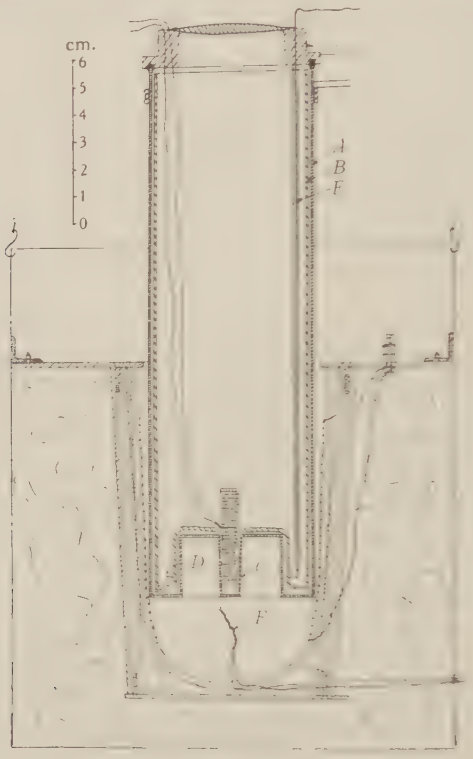

Figure 1. First type of discharge tube; vertical section.

temperature could be determined in this manner. Alternatively, a heating-current could be passed through it and its temperature could thus be varied independently of that of the furnace. The advantage of this second over the previous form of tube was that the region in which the discharge took place was uniformly heated with greater certainty; in addition, the heating was not effected by way of the material of the cathode, and so a suspected source of complications was eliminated.

§ 3. RESULTS

The structure of the bands is very simple, being similar, if we neglect the unresolved fine structure, to that for a $^1\Sigma \rightarrow {}^1\Sigma$ transition. The theoretical formula for the intensities in the R branch can in this case be shown to be*

$$I_{R(K)} \propto (K+1).\quad \exp\{-B_0' hc\,(K+1)\,(K+2)/k\tau\} \qquad \ldots\ldots(1),$$

* W. Jevons, *Report on Band Spectra of Diatomic Molecules* (Phys. Soc. 1932).

where $I_{R(K)}$ is the intensity of the Kth line of the R branch, K the rotation quantum number, B_0' the rotation constant for the upper $v' = 0$ level, and h, c and k have their usual significance. This expression neglects the variation of ν^4 in the small range covered by one band, where ν is the frequency of line*. Thus by plotting $\log I_{R(K)}/(K+1)$ against $(K+1)(K+2)$, we obtain a straight line with a slope inversely proportional to τ. Using the value 2·07 of B_0' as given by Jevons, and the value 0·69 for k/hc, we can calculate τ. Since the band shows the phenomenon of alternating intensities, each photograph can be made to give two values of τ by treating separately the strong and the weak lines. The line of closest fit was in each case estimated only, the number of points being insufficient to justify the use of

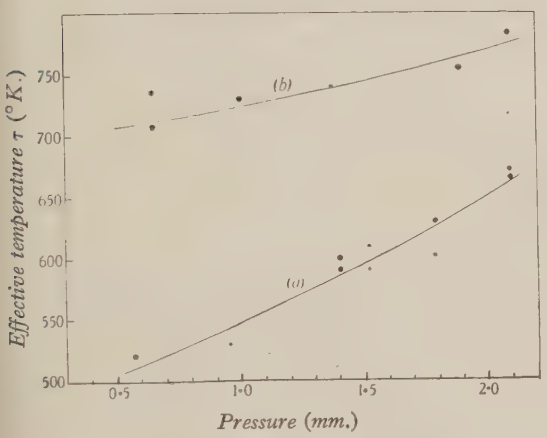

Figure 2.

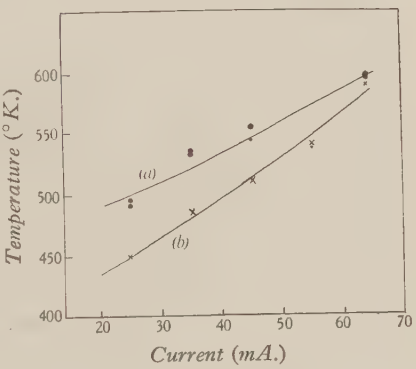

Figure 3.

Figure 2. Showing the effect of varying pressure on the effective rotation temperature; first tube. (*a*) With the tube at room-temperature; current 31 mA., potential between 424 and 487 V. (*b*) With the furnace-temperature equal to 650° K.; current 31 mA., potential between 386 and 450 V.

Figure 3. Showing the effect of varying current; first tube. Tube at room-temperature; pressure 0·72 ± 0·02 mm., potential increasing with the current from 386 to 446 V. (*a*) Effective rotation temperature. (*b*) Temperature given by thermocouple on cathode surface.

more elaborate methods. The accuracy varied greatly with the quality of the negative. A good result could be trusted to 2° or 3° K., while a photograph unevenly fogged or slightly scratched might give two values of τ differing by as much as 60° K. In the graphs reproduced herewith the size of the experimental point is intended as a rough indication of the reliability of the result, as judged by the deviations from the straight line in the original plot.

As a working hypothesis it was assumed that the following factors might conceivably affect the value of τ: (1) the shape, size, nature and relative position of the electrodes; (2) the pressure p of the gas; (3) the potential drop V across, and the current C through the tube; (4) the temperature of the gas. The first factor was eliminated by keeping the conditions in question constant in any one group of

* The error introduced by this approximation is of the order of 1 per cent of the values of τ and so does not affect the general conclusions.

results. With each type of tube they varied only slightly throughout the whole of the observations, and the different groups of results are, in fact, found to agree well amongst themselves. Moreover, the general conclusions are the same for both types of tube, the differences being small compared to the variations due to the other factors. Considering the vastly different design of the two tubes, we may say that the first factor produces only comparatively small variations in τ.

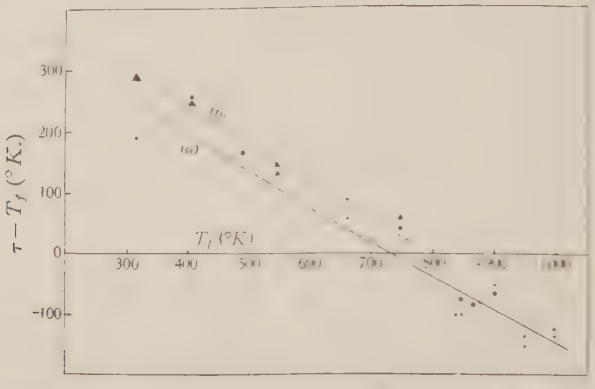

Figure 4. Showing the effect of varying furnace-temperature; first tube. Current 30 mA.; (*a*) pressure 0·64 ± 0·02 mm., potential between 378 and 412 V.; (*b*) pressure 1·90 ± 0·02 mm., potential between 406 and 436 V.

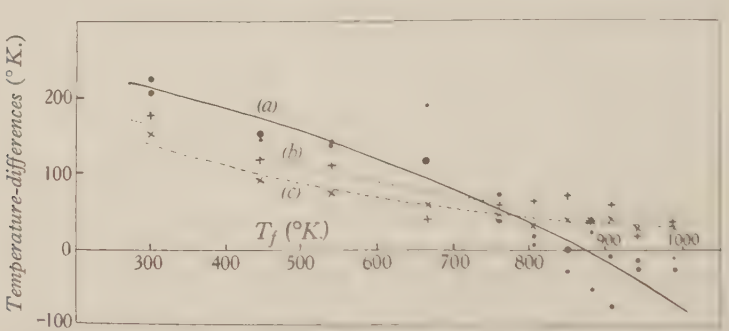

Figure 5. Showing the effect of varying furnace-temperature; second tube. Current 15 mA.; pressure varied, but density of gas approximately constant; potential between 301 and 351 V. (*a*) Effective rotation temperature, $\tau - T_f$; (*b*) temperature of cathode, $T_c - T_f$; (*c*) temperature given by resistance thermometer in negative glow, $\theta - T_f$.

The effect on τ of the other factors was found by varying each of them in turn, while maintaining the rest as far as possible constant. This could not be done exactly, since the variables are not independent: the temperature, pressure and current being fixed, the potential is determinate. Consequently, the first named three were treated as independent variables, and the value of the potential was noted in each case. Only a representative selection of all the results obtained is given here. Figure 2 shows that the effect of increasing the gas-pressure was to raise the value of τ, while figure 3 indicates that the same effect was produced by increasing the

current. These phenomena occurred whether the tube was surrounded by a water bath at room-temperature or heated in the furnace. Several series of results were obtained to show the effect of increasing the furnace-temperature, but one only for each type of tube is quoted, figures 4 and 5. To save space, the results are plotted to show the excess of the temperatures above that, T_f, of the furnace. In all cases τ increased with the furnace-temperature but at a lesser rate, becoming the smaller of the two at sufficiently high values. This holds both when the pressure, figure 4, and when the density, figure 5, of the gas is kept constant. Essentially similar curves were obtained with other constant pressures and currents, and with both types of tube. The temperature of the cathode increased with T_f but always remained the larger of the two, while the temperature indicated by the platinum resistance thermometer in the second tube was correlated with the cathode-temperature rather than with the effective rotation temperature or with the furnace-temperature, and was always slightly the smaller of the two.

T_f

§ 4. DISCUSSION

Considering first the temperature T_c of the cathode, we find that its behaviour is quite understandable. The cathode will be at the temperature of the furnace before the discharge is started, and subsequently it will be further heated by bombardment when a current flows. This is in agreement with the known fact that most of the energy input in a discharge tube is dissipated as heat at the cathode. It also accords with the increase in T_c with the current, recorded in figure 3 *b*. It is, moreover, to be expected that, with the second type of tube, the difference $(T_c - T_f)$ will become less as T_f increases. For if the heat were lost solely by radiation to the walls, the equation

T_c

$$a\,(T_c^4 - T_f^4) = \text{constant (energy input)}$$

would hold, while if the dissipation were solely by conduction along the leads, we should have

$$b\,(T_c - 300) = \text{constant},$$

300° K. being approximately the temperature of the cold end of the leads. Actually something between these two extremes will hold, and the shape of the curve 5*b* is in agreement with this scheme.

Turning to the temperature θ of the platinum filament, we note that had the latter been totally enclosed by the cathode we should have expected θ to be equal to T_c. In practice, radiation losses through the interstices of the cathode, and conduction along the filament leads, will both tend to make θ less than T_c, so that the general form of the relation is again explained. Thus it may be stated with a fair degree of certainty that the measured values of T_c and θ represent the real temperatures of the cathode and filament respectively. Consequently, in so far as a gas which is in neither thermal nor mechanical equilibrium can be said to have a temperature, that of the gas in the negative glow probably lies between the fairly narrow limits given by T_c and θ.

θ

What, then, is the significance of the quantity τ? Whether or not the conclusions of the last paragraph are accepted, τ is obviously not the gas-temperature, since it is most unlikely that this magnitude will be actually reduced by starting the discharge, and τ has been shown to be not only less than T_c but even less than T_f. Although it is not equal to the gas-temperature, its variation when the furnace is heated shows that it depends in part on this quantity. It is also probable that the increase in τ with increasing current is due indirectly to this same cause, since the increased cathode-temperature (see figure 3 *b*) will cause an increase in the gas-temperature in its neighbourhood. This view is supported by some further experiments with the second tube, in which a heating-current through the cathode raised both θ and τ, the increase being almost as much as when the cathode was raised to the same temperature by heating the furnace. The (τ, θ) relations for the two cases were very closely similar, but strict comparison is not possible since the conditions of pressure and current were not the same in the two cases. No such simple explanation is obvious to account for the variation of τ with the pressure, and the physical mechanism of this, as of the major problem of the divergence of τ from the gas-temperature, remains obscure.

In conclusion, it is of interest to consider the bearing of these results on the question of the duration of the excited state. They indicate that it is not long enough for the excited molecule to reach temperature equilibrium with its surroundings; for if it was, we should expect τ to be equal to the gas-temperature, i.e. approximately to θ. Thus formula (1), which is based on the assumption that such equilibrium is established, is not strictly valid. Van Wijk, in a recent paper*, has suggested that the life of the excited molecule is so short that the rotational-energy-distribution which existed before the molecule was excited is not appreciably disturbed before it radiates. In the Boltzmann factor in the formula for the intensity-distribution we must then insert the constant for the lower energy-level: if this is taken as being the ground state of the N_2^+ molecule, we obtain instead of (1)

$$I_{R(K)} \propto (K+1). \quad \exp\{-B_0''hc(K+1)(K+2)/k\tau\} \qquad \ldots\ldots(2)$$

where B_0'' is the rotation constant for this state. The effect of this change on our results is to reduce all the values of τ by about 7·5 per cent. If on the other hand we assume that the excited N_2^+ ion is formed direct from the ground state of the neutral molecule N_2, and if, following Van Wijk, we postulate that there is no change in the rotation quantum number when this takes place, B_0'' must take the value for the ground state of the neutral molecule. This would produce a reduction of about 4 per cent in the values of τ. If Van Wijk's suggestion represented the whole truth, the change should have made τ equal to θ, while, in actual fact, this is not true in either case, and the essential characteristics of the experimental curves remain unaltered and unexplained.

Thus neither alternative holds good, and all that can be said at present is that the rotational-energy-distribution is modified when the molecule is excited. The resulting energy-distribution is approximately of the Boltzmann form, but the

* W. R. Van Wijk, *Z. f. Phys.* **75**, 584 (1933).

effective temperature deduced is not identical with the gas-temperature; it can, in different circumstances, be either greater or less than this quantity, and depends on some other factors which have not yet been taken account of in the theory.

§ 5. ACKNOWLEDGMENTS

Finally, I must tender my very best thanks to Dr Milner and his staff in the University of Sheffield for their assistance and encouragement during the progress of the work, and to the Department of Scientific and Industrial Research and the Sheffield Town Trustees for the scholarships which rendered it possible.

DISCUSSION

Mr J. H. Awbery. The fact that the effective temperature exceeds the furnace-temperature by smaller amounts as the latter is raised, until their relative positions becomes actually reversed, is surprising. Is it possible that this result is due to errors in assessing the furnace-temperature? The latter was measured by means of a copper-constantan couple, an instrument which is not usually found reliable at temperatures of the order of 1000° K., where oxidation of the copper limb is liable to become serious. Possibly the author could show the calibration of the couple before and after use, and so dispose of this doubt.

Author's reply. I was unfortunately not able to recalibrate the thermocouple after use, so it must be admitted that Mr Awbery's suggestion is not altogether impossible. Nevertheless I do not think that it is probable, since the couple, when heated, would be in an atmosphere of almost pure nitrogen. Oxidation may have occurred during the calibration, but then this would be taken account of in the results obtained. Moreover, results consistent within the limits of error were obtained independently of the previous heat treatment of the couple.

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THE MEASUREMENT OF FLAME TEMPERATURES IN A PETROL ENGINE BY THE SPECTRAL LINE-REVERSAL METHOD

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AND
B. J. LLOYD-EVANS, M.Sc. (Eng.), M.I.Mech.E.,
both of the University of London

Communicated by Dr L. F. Bates, January 19, 1934. *Read March* 16, 1934.

ABSTRACT. Hitherto no satisfactory method has existed for the measurement of the temperatures which occur during combustion in a petrol engine, but it is shown that the reversal of a spectral line provides a suitable method. It is found that the maximum temperature in the engine cylinder persists for a longer period than the maximum pressure.

§1. INTRODUCTION

One of the earliest attempts to measure flame-temperatures was that of F. Kurlbaum*, who compared the temperature of a luminous flame with that of a black body. In the case of a non-luminous flame, Féry† proposed to introduce a suitable metallic salt into the flame and to apply the principle of spectral line-reversal. This was first enunciated by Kirchhoff and the method consisted in obtaining a luminous source at the same temperature as the flame under observation and then measuring the temperature of this source by means of an optical pyrometer. E. Bauer‡ showed that the result was independent of the choice of metallic salt.

This has been confirmed and the method more fully developed in a valuable paper by E. Griffiths and J. H. Awbery§. In that paper many of the possible sources of error have been discussed and in particular the method has been applied to the measurement of the temperature of combustion of carbon monoxide and air in an explosion vessel.

In this communication, the authors have utilized the method for the determination of the flame-temperatures at various points of the working cycle of a petrol engine, these temperatures ranging from 1200° C. to 2200° C. This is a problem which has exercised the minds of engineers for some years and it is felt that the results are not without interest to physicists. Previous experimenters have almost

* *Phys. Z.* **3**, 187 (1902).

† C. Féry, *Compt. Rend.* **137**, 909 (1903).

‡ *Compt. Rend.* **148**, 908 and 1756 (1909). Cf. *Traité de Pyrométrie Optique* by G. Ribaud. pp. 415, 416 (1931).

§ *Proc. R.S.* A **123**, 401 (1929).

invariably made use of thermocouples; these have inherent disadvantages, such as time lag and, in the case of very small and unprotected couples, instability of calibration*. A further difficulty in the use of thermocouples is that in an engine they are subjected to violent explosion waves, which cause mechanical breakage, even when the maximum explosion temperature is reduced below the fusing point of the couple by diluting the explosive mixture present. The highest temperature so far recorded in a gas engine was computed solely from the fact that the end of the couple had just fused†. Other experimenters‡ on gas engines have used resistance thermometers, but these again could not be employed at the highest working temperatures. Since the commencement of these tests, it has, however, come to the authors' notice that a couple, which is said to be satisfactory up to a temperature of 2000° C.§ has been devised.

§2. APPARATUS

As a full description of the principles involved is given in the paper of Griffiths and Awbery||, the present authors will limit themselves to describing the apparatus used in their own tests.

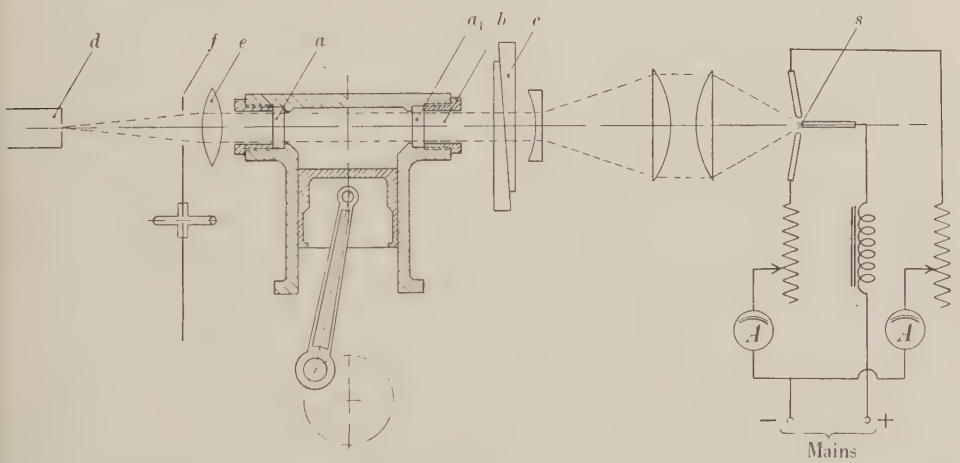

Figure 1. a, a_1, quartz windows; b, parallel beam; c, neutral wedges; d, spectroscope; e, lens; f, stroboscopic disc; s, source of light.

In figure 1, a and a_1 represent two quartz windows let into the combustion space of an engine cylinder designed specially to receive them. Light from a source s passes through a combination of lenses and emerges as a parallel beam at b, the intensity being controlled by a pair of adjustable neutral wedges at c. After passing through the engine cylinder, this beam is focused on to the slit of a spectroscope d by means of the lens e. Between this lens and the spectroscope is arranged the

* Cf. "Pyrometric Practice," *Bureau of Standards Tech. Papers*, **170**, 23 (1921).
† E. Coker and W. A. Scoble, *Proc. Inst. C.E.* **196**, 1 (1914).
‡ H. Callendar and W. E. Dalby, *Proc. R.S.* A **80**, 57 (1908).
§ O. Feussner, *E.T.Z.* **54**, 155 (1933).
|| *Proc. R.S.* A **123**, 401 (1919).

stroboscopic disc *f* which is driven at half the speed of the engine so that light passes to the slit in *d* once in every working cycle. The disc can be phased to any desired position of the engine-mechanism. The wedges *c* are adjusted so that reversal of the sodium line occurs while the engine is running. The engine is then stopped and the spectroscope replaced by a monochromatic pyrometer of the disappearing-filament type. In order to obtain a reading on the pyrometer, the left-hand quartz window *a* and the lens *e* are removed.

For each point three sets of readings were taken after the conditions of running, such as load, temperature of cooling-water, etc., had become steady. As this necessitated a certain lapse of time while adjustments were made, it was found desirable to use a steady source of light at *s*. A standard Pointolite lamp, although it acted as a constant source of light, did not allow wedges of reasonable density to be used, and a serious time lag was introduced if an attempt was made to vary the intensity of light by means of a series resistance in the Pointolite circuit. The authors therefore used an arc lamp provided with one positive and two negative carbons, each 6 mm. in diameter, as described by Forrest* who used this lamp as a standard of illumination.

It was found that reliable readings could be obtained with the engine-crank placed at settings between 5° and 70° later than that with the piston in its uppermost position, known as the "top dead centre." Over this range it was possible to repeat individual readings to within 15° C. Readings taken after this range of settings became less certain since the temperature was then about 1200° C. and the spectrum observed under these conditions approached the limit of visibility. It was possible to improve matters in this respect by an alteration in the lens system or by increasing the size of the aperture in the stroboscopic disc, but it was found inconvenient to work the alternative-lens method as slight alterations in alignment, due to vibration of the engine, were liable to cause errors. Again, an increase in the size of the hole in the disc would make the cyclical position at which temperatures were recorded less definite.

Measurements taken before top dead centre was reached were liable to slight errors, firstly because in this region combustion was not general but very localized, and secondly there were variations between one explosion and another, which are inevitable with this type of engine. It is well known that pressure-measurements in a petrol engine at individual points near the maximum, taken with a Farnboro indicator, also show similar marked variations.

The authors used a Hopkinson optical indicator to obtain photographic records of the (pressure volume) diagrams, as shown in figure 2. The complete data are collected in figure 3, which shows pressure plotted on a base of time or crank-angle, together with the corresponding temperature-readings. The ignition advance for both curves was 20°, which was found to be the best setting for this particular speed (1016 r.p.m.) at full load.

As has been pointed out by Griffiths and Awbery† the method is only valid if

* J. F. Forrest, *Electrician*, **71**, 729 (1913).

† *Proc. R.S.* A **123**, 401 (1929).

the engine flame is non-luminous. With the petrol used, Pratts No. 2, no continuous spectrum could be observed when the centre of the cylinder-head was focused on the slit of the spectroscope. To reduce any possible error in case of the flame being slightly luminous, the lens system was arranged so that very little light from the gas itself reached the spectroscope; consequently the effect on the continuous spectrum from the arc lamp was negligible.

The amount of sodium normally present in the flame of a petrol engine is insufficient to give really satisfactory reversal. A 1 per cent solution of sodium ethylate in alcohol was therefore added to the petrol in the proportion of 15 cm^3 to each gallon. This fluid is combustible and appears to have no measurable effect on the performance of the engine.

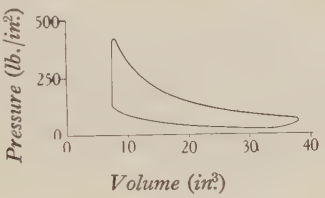

Figure 2.

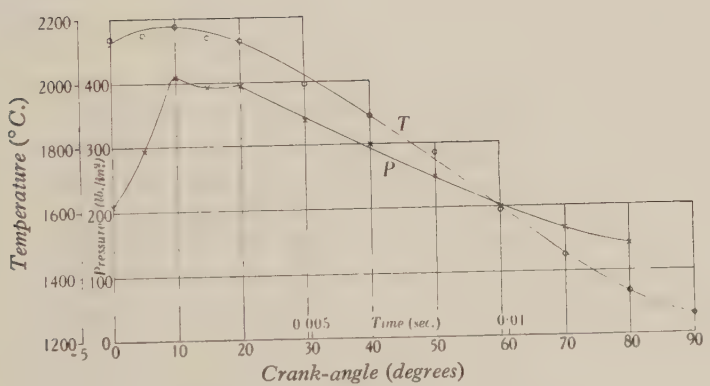

Figure 3. *P*, pressure; *T*, temperature.

§3. RESULTS

It is of interest to note that in figure 3 the period of maximum pressure is of much shorter duration than that of maximum temperature. Probably the explosive mixture burns comparatively slowly so that various parts of the charge in turn rise to the highest temperature, while the pressure still continues to rise until a considerable proportion of the gases has been ignited. There appears to be a limit (possibly due to dissociation) to the highest temperature as has been pointed out by

Tizard and Pye*, who have made a number of calculations of explosion temperatures in which this factor is taken into account. A direct comparison with their calculations cannot be made, however, as they assumed that all the charge was burned instantaneously at the top dead centre. For heptane, for example, with the chemically correct proportion of air, they calculated a maximum temperature of 2590° C.

The present authors consider it unnecessary to include here a more complete analysis of their results which bear on the engineering side of the problem; these will, it is hoped, be reported elsewhere. In conclusion, they wish to point out that this new development of the spectral line-reversal method should prove of considerable value in the investigation of the problems of the internal-combustion engine.

§4. ACKNOWLEDGMENTS

The authors wish to thank Professor E. N. da C. Andrade for the loan of a direct-vision spectroscope and Dr L. F. Bates for many useful suggestions.

DISCUSSION

Lord Rayleigh remarked that the use of an arc lamp with one positive and two negative carbons for producing steady illumination had been attributed by the authors to Forrest (1913). It had, however, been adopted as early as 1908 by Sir William Abney at the Imperial College of Science.

Dr L. F. Bates. I was very pleased to be able to communicate this paper, because it describes an interesting and important practical application of some fundamental experiments carried out by two of our most active Fellows, Dr Griffiths and Mr Awbery. I have followed the work which has just been described from its commencement, and I have never ceased to be impressed by the speed and accuracy with which the individual determinations of the temperature of the gas at any instant can be made. It will have been noted that these individual readings can be repeated to within 15° C. This does not represent the limit of accuracy of the temperature-measurements themselves, but is merely a statement of the fact that the working conditions of the petrol engine vary to this extent, for obviously the supply of petrol etc. cannot be strictly controlled. I may add that considerable interest is being taken in these measurements by engineers. Only a few days ago a foreign engineer, reading a paper before a London institute, stated that there was no method of measuring the temperature inside the cylinder of a petrol engine with any degree of accuracy. I hope that the advance proof of this paper will give him satisfaction.

Dr Ezer Griffiths. The authors' method of selecting the point at which readings are to be taken is simple and effective. The temperature measurements are free from time lag, but can the same be said of the pressures?

* *Report Empire Motor Fuels Committee, Inst. Auto. Engrs.* **18**, 1 (1923).

Mr J. H. Awbery. Since the measurements made on this engine include simultaneous determinations of the pressure p, the temperature T, and the specific volume v, it is possible to substitute in the equation $pv = nRT$ and so determine n at different points of the cycle. This should throw some light on the questions whether dissociation takes place, and whether after-burning is appreciable. The fact that the flame is radiating cannot be taken as evidence that combustion is still proceeding, and therefore the temperatures alone do not give evidence as to after-burning.

A word of warning may be uttered in connexion with the determination of n in the above equation. The experiments give the average values of p and of v taken through the cylinder at any instant, but the value of T is the maximum. If, for example, the centre of the charge is the hottest part, then it is this temperature that is measured. Thus n may have relative significance—changes of n while the valves are closed would be evidence on the chemical questions involved—but the actual value is not accurately known.

Authors' reply. We have to thank the President for calling our attention to the fact that the three-carbon arc was in use as early as 1908; we also thank Dr Bates for his observations.

With regard to the question of time lag in the pressure-measurements, the Hopkinson indicator used was the best instrument available and should be reasonably accurate at the speed of the engine, 1016 r.p.m. We had not overlooked the suggestions made by Mr J. H. Awbery, and in fact we have already found that pv/T, calculated for a number of points on the expansion stroke, is far from constant. The value increases with time or increase of the volume v behind the piston, and probably implies that combustion is still proceeding.

551.585.2

AN APPARATUS FOR THE DETERMINATION OF THE DEW POINT

By ERIC B. MOSS, B.Sc. (Lond.)

Received January 24, 1934. *Read and demonstrated March* 16, 1934.

ABSTRACT. The paper describes an optical system which, using diffraction by the dew droplets on a mirror, aids greatly the visual detection of dew formation. Then follows an account of the application of this system to an automatic photoelectric apparatus for maintaining a mirror at the dew point. Results are given of experiments made to find the effect on the accuracy obtainable of the difference between atmospheric temperature and the dew point, and of varying air velocities round the apparatus.

§ 1. INTRODUCTION

There are several physical methods of finding the humidity or partial pressure of water vapour in an atmosphere, but the greatest reliance is usually placed on the dew-point determination even though the measurement may call for skill on the operator's part. Suggestions* have been made of arrangements and devices designed to aid the detection of dew and thus to increase the precision of measurement, and the first part of this paper is devoted to the description of an optical system for this purpose.

The eye sees dew on a mirror by reason of light which is diffracted and scattered by it; therefore to secure the greatest sensitivity not only must the mirror be illuminated intensely but also the eye must be shielded from the directly reflected beam so as to be in the most sensitive condition. Such shielding requires the reflected beam to be definite, and the use of an optically worked mirror in the system under discussion ensures this. Further, this system uses the relatively intense diffraction which occurs at small angles to the direct beam. The design is well adapted for use with photoelectric apparatus, which is not called upon to respond to small changes in an intense beam, as it would be if the fall produced in the reflecting-power of the mirror by a dew film were used as the basis of detection. Instead the photoelectric apparatus only has to detect the incidence of light, and thus it can be used at high sensitivity.

§ 2. THE OPTICAL SYSTEM

Recently the author devised a sensitive apparatus for the detection of smoke† and since it depended on diffraction of light by the smoke particles it seemed capable of being adapted to detect dew on a mirror. Alterations were necessary, and the modified optical system is shown in figure 1.

* E. Griffiths, "Discussion on Hygrometry," *Proc. Phys. Soc.* **34**, 27 (1921); S. R. Williams, *Rev. Sci. Inst.* 4, 549 (1933).

† British Patent Specification No. 396390.

The mirror *C*, figure 1, is that on which the dew is formed, and in the present apparatus is of stainless steel, optically worked and polished. It is illuminated by a beam from the lamp, concentrated by the condenser *A* which forms an image of the filament near lens *B*. This lens is of wide aperture, and of such a focal length that an image of lens *A* is formed on the stop *D*. Since all light entering the system does so through this lens *A*, stop *D* will intercept that light-reaching lens *B* by direct reflexion. Lens *B* also produces an image of the mirror *C*, but this is larger and beyond the stop *D*, at *E*, and the rays forming it are not obstructed seriously by the stop. As long as the mirror is a perfect reflector the image will be very faint, but when dew is deposited the mirror and its image become luminous. For observation purposes it is seen conveniently by a low-magnification eyepiece at *E*.

If it were assumed that the only difference in diffraction-effect between particles on a mirror and particles in suspension is that the former will produce twice as great an intensity as the latter, then it would be possible to calculate the distribution of

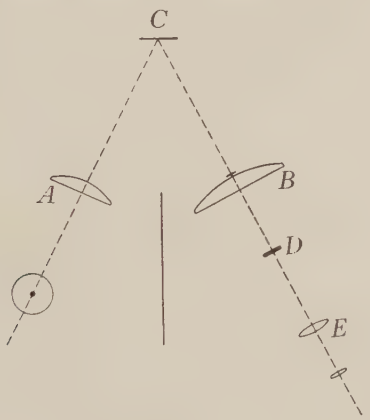

Figure 1. Diagram of optical system.

the diffracted light. This would require a knowledge of the particle-size, but since this can most easily be measured indirectly, by ascertaining the light-distribution, and since such an experiment will give the information required directly, there is little to be gained by a theoretical investigation.

The chief experimental difficulty is the instability of a dew film which precludes measurement of intensity at many points with one of the many instruments available. The difficulty was overcome by the following method which explored the whole field of 180° in front of the mirror in 2 seconds.

A mirror, fixed in the side of a glass tube so that it could be cooled with ether, was mounted vertically. Below it with its axis on the vertical diameter of the mirror was a spindle carrying an arm about 150 cm. long. A plane mirror, mounted at the outer end of this arm, reflected light falling on it from the fixed mirror on to a photocell. The latter mirror was illuminated by a beam having an aperture of 5°, and the moving mirror also subtended a similar angle at the fixed one. The shaft made a complete revolution in four seconds, and as it turned the scanning mirror

explored the field of light reflected and scattered by dew on the other mirror. The photocell was used with a single amplifying valve having the string of an Einthoven galvanometer so connected in the anode circuit that the deflections were proportional to light incident on the photocell. Therefore, at any instant the position of the string was a measure of the light leaving the fixed mirror in the direction of the scanning mirror at that moment. The movements were recorded photographically in the form of a (current, time) curve, which was readily converted into a polar-intensity curve from a knowledge of the angular velocity of the scanning mirror.

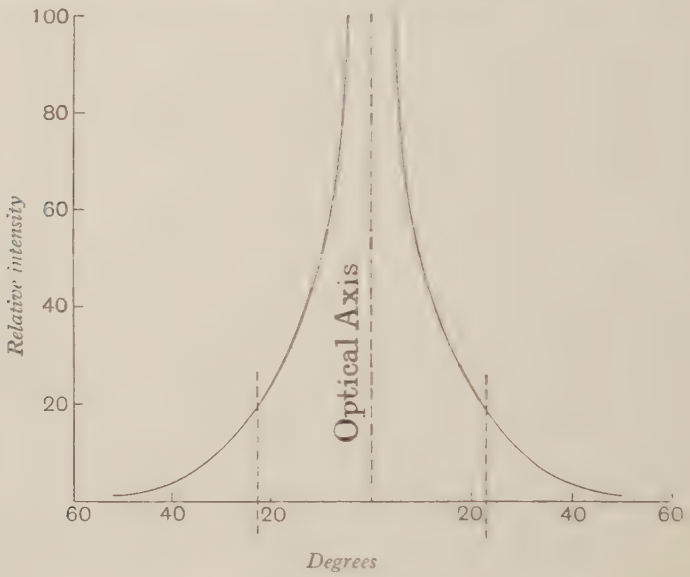

Figure 2. Intensity-distribution from dew-covered mirror.

The result for a wave-length range of 0·6 to 0·8μ. with the incident beam normal to the mirror is shown by the graph in figure 2. This shows that most of the diffracted light leaves the mirror within a total angle of 45°, and lens *B*, figure 1, in the optical system has been so arranged as to subtend approximately this angle at the mirror.

§ 3. THE AUTOMATIC DETERMINATION OF DEW POINT

The usual procedure for the estimation of the dew point is to permit air to bubble through ether, thereby cooling a surface on which the formation of dew is looked for. When it is observed the air flow is stopped, and the ether-temperature is noted when the last trace of dew disappears. This cycle is often repeated to confirm the first result. In the automatic determination the cycle is repeated continuously at intervals of from three to ten seconds depending on the dew point, air flow and other variable conditions of experiment. The temperature of the ether then oscillates only a fraction of a degree on either side of the true dew point, and the mean temperature, as will be shown later, is in very close approximation to it.

The eyepiece at *E* is now replaced by a photocell which is connected to a single-valve d.-c. amplifier. When the anode current exceeds a certain predetermined value it operates a relay which in turn closes a solenoid air valve, shutting off the air stream which produces a cooling effect by bubbling through ether or other suitable liquid. This is contained in a thin-walled brass tube, thoroughly lagged with asbestos except where the tube is cut away to receive the optically worked steel mirror. This is soldered on. The ether is thus in direct contact with the back of the mirror, which is less than 1 mm. thick. Stainless steel was chosen as the best material for the mirror on account of its thermal conductivity and its ability to take and retain a good optical polish.

It is essential that the supply of ether or other refrigerant be steady and the level constant, so a supply of about 150 cm^3 is contained in a small tank which feeds the apparatus through a float chamber, as in a carburettor. Under average conditions this quantity lasts for several hours.

The air valve is of simple construction, being an ironclad solenoid with a moving plunger of mild steel fitted with a stainless-steel needle which comes into contact with a gun-metal seating. A current of 0·15 A. at 2 V. will close it against a pressure of 10 lb./in^2. As this valve is only an on-and-off control, a small tap is provided to adjust the flow.

In the apparatus which is shown, the gas-filled lamp has a rating of 24 watts at 12 volts. It has a screwed cap and the filament is vertical. Lenses *A* and *B* have focal lengths of 6·0 and 5·0 cm. with diameters of 2·2 and 5·0 cm. respectively. The distance from the mirror to both *A* and *B* is 9·0 cm. The lamp, lenses, valves, photocell and relay are enclosed in a metal box measuring 23 cm. × 14 cm. × 17 cm. which has an elongated aperture to pass the incident and reflected light. The tube with the mirror is carried at the end of a hollow arm, through which the ether is fed, by a ball joint permitting adjustment to the correct position. The ether-feed device and air valve also are secured to the base of the box.

The electrical circuit is shown in figure 3. It contains no novelties, and the only point worthy of note is the method of obtaining a current to oppose the steady anode current, so that it is possible to work on the steep part of the valve characteristic. It will be obvious that a current will flow through the relay coil unless the anode current flowing through *R* produces a p.d. equal to 2 volts. When the potential exceeds this value the relay is deflected in the direction which closes the circuit through the solenoid. To protect the somewhat delicate contacts of the relay from sparking an inductive surge-absorber consisting of a single-disc copper-oxide rectifier is used. The valve is of the 2-volt power type with a mutual conductance of 3·85 mA./V., and a standard photocell of talking-picture type is used with an applied potential of 80–90 V.

When the apparatus is working the air flow is intermittent, and to avoid excessive hunting the frequency of intermission must be as great as possible and this is secured by so adjusting the grid-bias that the relay trips with the minimum of dew. For example, if the relay will make contact when the anode current exceeds 5 mA. the grid-bias should be adjusted to make the anode current about 4·8 mA. with a

clear mirror. As the mutual conductance of the value is 3·85 mA./V. this means that the relay is being operated by a photoelectric current of $5 \cdot 10^{-9}$ A., and the dew producing this will be barely visible to a human eye observing it directly, although it can easily be discerned by the optical arrangement described earlier.

For instantaneous temperature-readings a mercury thermometer is quite suitable, but a platinum resistance thermometer offers the possibility of remote reading and recording. A thermocouple may be used, but a single junction would not give quite the same accuracy as either of the other two alternatives.

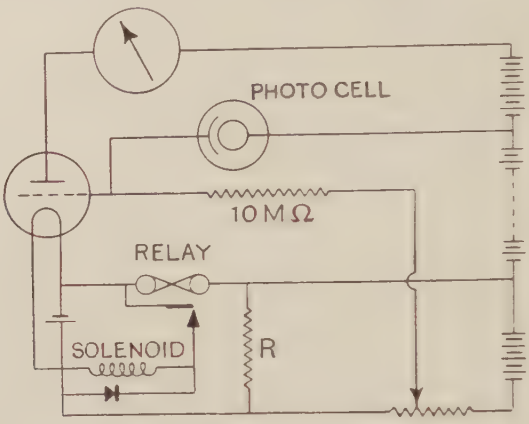

Figure 3. Electrical circuit.

§ 4. EXPERIMENTAL

The steel mirror is approximately 1 mm. thick, this being the minimum thickness with which optical perfection can be ensured, and it seemed that there must be a temperature-gradient in material of this thickness. To measure such a difference directly would involve the evaluation of small quantities, but an indirect measurement is feasible. With the correct manipulation or control, the outer surface is maintained at the dew point. If it is assumed that a difference of temperature exists between the inside and outside of the mirror, then it is logical to expect an increasing difference with rising air-temperature if the dew point is kept constant. Consequently, when the apparatus is working in an atmosphere having a constant water-content, a rise in the air-temperature will be accompanied by an apparent fall in dew point as shown by the ether-temperature. Experiments were made along these lines to determine the temperature-difference indirectly.

The tube with the mirror was placed behind a window in a miniature wind tunnel, so that air which was driven through the tunnel passed parallel to the plane of the mirror. At the inlet end the air was blown over wet- and dry-bulb mercury thermometers. These were shielded from radiation from a wire resistance mat, which farther up the tunnel provided the means for raising the air-temperature without affecting the water-content or its determination.

Early tests on the automatic apparatus showed that it did not work satisfactorily in a still atmosphere, and one purpose of the wind-tunnel tests was to find the effect

which air-velocity had on the readings. The same apparatus was employed, the velocity being measured with a Pitot tube in conjunction with a Threlfall gauge. It was convenient to make the two sets of experiments concurrently as the air-temperature and velocity could be changed very easily; thus for any particular dew point a number of readings were obtained for a range of temperatures and velocities. Some results are shown in figures 4*a* and 4*b*.

Figure 4*a* shows that considerable errors may be expected at low air-velocities, that is, at velocities of less than 200 cm./sec., and the points in figure 4*b* were all determined above this value. This latter diagram seems to deny the existence of

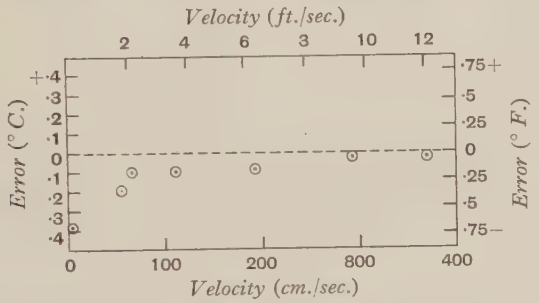

Figure 4*a*. Instrument error at different air velocities.

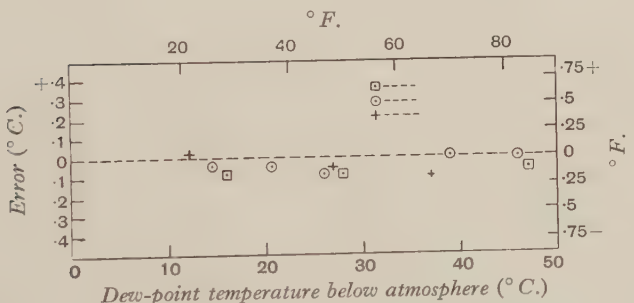

Figure 4*b*. Instrumental error at different dew-point depressions.

any temperature difference between the sides of the mirrors, since the error is not proportional to the difference between apparent dew point and air-temperature. In view of the constant discrepancy between the true and apparent dew points the thermometers used were rechecked carefully against fixed points and National Physical Laboratory standard without the disclosure of any explanation. Probably we must look for this to figure 4*a* which shows a decreasing error with rising velocity; it is probable that at still higher velocities the error would vanish. It may be due to a relatively stationary layer of air in contact with the mirror. Unfortunately apparatus was not available to test the validity of this inference.

That the error is due to absorbed radiation from the lamp is denied by the way in which the dew forms and evaporates. The centre of the mirror on which the light falls is the first part to be covered and the last from which the dew vanishes.

During these experiments the air flow through the ether with the solenoid valve open was the maximum that could be passed without liquid ether being bubbled out of the tube. This flow permits dew points 50° C. below air-temperature to be reached, but gives some hunting at low depressions. For the best results the air flow should be adjusted to the minimum which will give the necessary depression with certainty.

It may be concluded from the results that this apparatus permits the determination of the dew point with a possible error not greater than 0·1° C. provided that the air-velocity parallel to the mirror is greater than 200 cm. sec. and that the electrical controls are correctly adjusted.

§ 5. ACKNOWLEDGMENT

The instrument has been developed and used in the laboratory of the Cambridge Instrument Co. Ltd., and it is to the Directors of the Company that I am indebted for permission to publish this description.

DISCUSSION

Lord RAYLEIGH suggested that an optical system similar to that adopted by the author might be used for photographing finger-prints or for intensifying weak negatives. An image of the surface bearing the finger-prints or of the silver deposit in a negative could be formed by scattered light alone, the direct light being eliminated as in the author's apparatus.

Mr J. GUILD. It seems desirable, now that methods of ever increasing precision are being developed for measurements of this kind, to enquire to what accuracy the assumptions underlying the method are valid. In the original Regnault dew-point apparatus, of which these photo-electric devices are simply refinements, the fundamental assumption is that the temperature of the metal surface, when dew begins to form, is the temperature at which the air in its immediate neighbourhood is just saturated by its moisture content. No doubt this assumption was true enough to the kind of accuracy attained by the original dew-point apparatus, but it is by no means obvious that it is absolutely true. The formation of dew-drops on a highly polished surface is probably not a simple phenomenon, and it is quite possible that a certain degree of supersaturation is necessary before deposition begins. What is the existing evidence on this point? It is clearly necessary before we can usefully advance the precision of dew-point measurements to establish without doubt that the temperature at which dew condenses on a polished surface does not differ from the saturation temperature by an amount comparable with the precision of the instrument.

Dr L. SIMONS. Does the author find that the mirror retains its initial polish? Does not dust, having the same optical effect as dew, collect upon it in time?

Dr EZER GRIFFITHS. The method described for projecting the image of the surface on which dew is deposited is neat and should prove of practical value in

cases where the surface has to be situated at some distance from the point of observation. I used the photoelectric cell as a means for detecting dew some five years ago. The description of the arrangement will be found in a special report of the Food Investigation Board entitled "The measurement of humidity in closed spaces."

Mr R. W. Powell. I should like to know if the author has compared the value of the dew-point temperature as recorded by this automatic instrument with that obtained by the normal visual method. Such data would be of particular interest since it would appear that the automatic instrument tends to give somewhat low values.

With regard to the temperature-drop through the mirror itself the author states that "stainless steel was chosen as the best material for the mirror on account of its thermal conductivity." Just what is meant? The thermal conductivity of stainless steel is low compared with that of most metals. However, this is not actually of vital importance since the temperature-drop through the mirror in 'still' air conditions should not exceed 0·01° C. when the depression is 20° C. I assume that the experimental test was really made to determine if a temperature-discontinuity existed between the metallic surface and the cooling-liquid in which the thermometer was immersed. In view of the possibility of such a discontinuity I submit that it would be an improvement to measure the dew-point temperature by means of a thermocouple soldered to the metallic surface.

I should also like to ask how the true dew-point was determined, and to point out that if wet- and dry-bulb thermometers were suspended in the wind stream as described, the true maximum depression of the wet bulb would only be attained at the higher wind-speeds.

Author's reply. In reply to Lord Rayleigh: A photometer based on the suggestion of measuring the light scattered by the silver particles has been constructed by Brentano, but he used a stop in a position equivalent to one immediately in front of lens *B*, figure 1. If the stop is used here it ignores the possibility of scattering from the surface of lens *A*. In practice, this cannot be ignored and the position of stop *D* shown in figure 1 is theoretically the most sound.

In reply to Mr Guild: There is no evidence in these experiments that the mirror surface is at the saturation temperature of the air when dew is formed. If it is not so then the resultant error is included in that shown in figures 4*a* and 4*b*.

It is found, as suggested by Dr Simons, that some dust does collect on the mirror, but the rate at which it is deposited seems to bear some relation to the density of the dew-deposit which is allowed to form. This apparatus works with a small density and although as a result it is sensitive to a minute dust-deposit, it is found that the effect of dust on the mirror is less serious with small dew-deposits. For the best working it is desirable to clean the mirror each day, and this necessity for frequent cleaning was one reason for the choice of stainless steel as the material for the mirror.

I am interested to note from Dr Griffiths's remarks that a photoelectric method has been in use for the detection of dew-deposit at the National Physical Laboratory. The vital difference between the two systems, however, is that this one utilizes the intense scattering which occurs at small angles to the truly reflected beam. More light is thus available for the operation of the photoelectric device from a given deposit of dew in this apparatus than in the one used at the National Physical Laboratory.

In reply to Mr Powell: The readings of this instrument were not compared directly with a normal manually operated visual instrument, but figures 4*a* and 4*b* show the difference between the readings given by the apparatus and the dew point as calculated from wet- and dry-bulb thermometer readings with the aid of the tables published by the United States Department of Agriculture Weather Bureau, and I have always found that the normal visual method gives results in agreement with these tables. A compromise had to be made in the choice of material for the mirror. It was necessary to choose a metal which would take and retain a good optical polish. In view of the need for high thermal conductivity, stainless steel was chosen. The results given in figure 4*b* show the magnitude of the instrumental error for various differences between dew point and atmospheric temperature. It is sufficiently small to be negligible for most humidity-measurements. If extreme accuracy is required then allowance can be made. The difficulties of fixing a thermocouple to the mirror surface so as to obtain the surface temperature would be great, and as its presence would hinder the cleaning of the mirror there seems to be no reason for thinking that it would give improved results. In checking the instrument, the effective air-velocity for the wet bulb was of course not overlooked, and correct conditions were maintained.

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HUE-DISCRIMINATION IN NORMAL COLOUR-VISION

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ABSTRACT. Hue-discrimination curves have been obtained for five normal observers; the apparatus and method of observation are described and the results discussed. Two minima in the discrimination curve are found at about 0·60 μ. and 0·49 μ. and a secondary minimum at 0·45 μ. The curves are appreciably different from those normally reproduced in text-books, particularly at the red end of the spectrum where a secondary minimum has generally been shown. An analysis of the work of previous investigators, however, shows that the evidence for this secondary dip is very inconclusive.

§ 1. INTRODUCTION

One of the most significant characteristics of colour-vision is the power of distinguishing between the colour sensations evoked by different spectral radiations. The crude division of the spectrum into red, orange, yellow, green, blue and violet, is well known, but a more exact analysis of the colour-differences of neighbouring radiations is also possible. For instance, if we view two patches of light in a photometric field, one illuminated by a radiation of wave-length λ and the other by a slightly different radiation, of wave-length $(\lambda + \delta\lambda)$, then as $\delta\lambda$ is increased the two halves of the field become just noticeably different in colour. (It is important that the intensities of the two patches be kept equal, to ensure that only the hue-discrimination function is acting.) The value of $\delta\lambda$ will vary with λ and a curve can be obtained relating the two quantities. This curve is generally known as the hue-discrimination or hue-sensibility curve and most probably is closely related to the fundamental response or excitation curves. Its evaluation is therefore of considerable theoretical interest, as well as of practical importance in colorimetric work.

In the course of other investigations the authors had occasion to determine the hue-discrimination curves for their own eyes and were surprised to find that although the two curves obtained agreed quite well with each other, yet they were markedly different from the average curve usually reproduced in text-books or analysed in modern discussions of visual theory. The frequency with which the curve is referred to is a sufficient indication of its importance and the need for its reliable determination, and it was therefore decided to investigate the matter more fully.

The practical value of the data is that they enable us to know the significant accuracy of colour-measurements or colorimetric calculations and to measure the visual characteristics of an observer and to assess his value for work involving colour-discrimination, while their theoretical value lies in their forming a basis for the development of a satisfactory theory of colour-vision, for either the normal, the anomalous or the colour-blind observer. For these purposes there would not appear to be any call for the determination of a standard curve on the lines of the standard luminosity curve or the standard colour-mixture curves. It was therefore considered the better procedure to determine curves for a few observers only, under properly controlled conditions and with the necessary practice and adequate checking of the results, rather than to obtain a greater number of curves with the possible sacrifice of individual reliability.

§ 2. APPARATUS AND TECHNIQUE

The apparatus used was the Wright colorimeter, slightly modified to meet the special requirements of the experiment. A diagram of the optical system is shown in figure 1. Light from a pointolite lamp is focused on the slit S and made parallel by the collimating lens C. This parallel beam, after passing through a right-angled

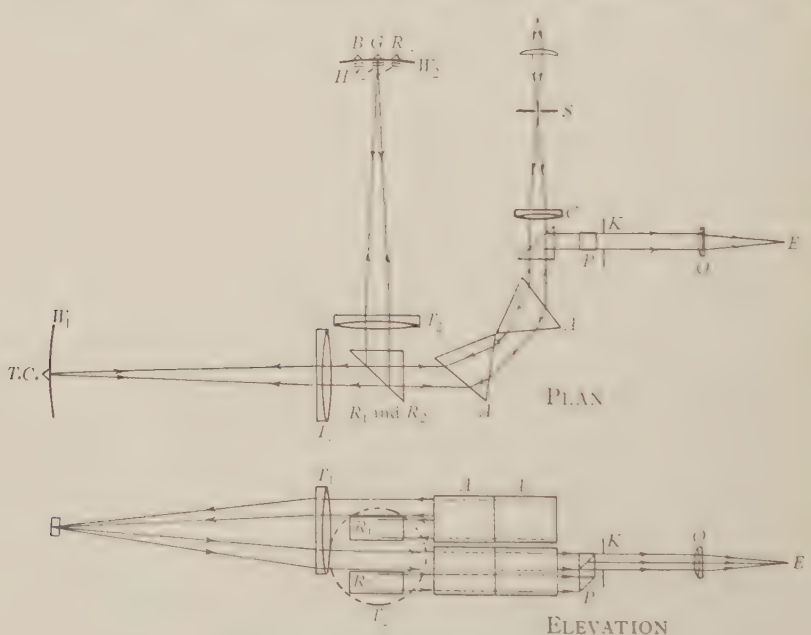

Figure 1. The Wright colorimeter.

prism, passes through the dispersing prisms A. The top half of the dispersed beam passes over the top of the right-angled prisms R_1 and R_2, through the lens T_1, and is brought to a focus in the form of a spectrum at W_1. The lower half of the dispersed beam is reflected at 90° by the prism R_1, passes through the lens T_2, and

is brought to a focus in the form of a spectrum at W_2. Three portions of the spectrum W_2 are reflected back by right-angled prisms *R*, *G* and *B*, and return through the optical system to the right-angled prism *D* where they are reflected on to the photometer prism *P*. When the instrument is being used as a colorimeter, these portions of the spectrum form the primaries, but in the present research they were used, singly, to form the comparison source. Any portion of the spectrum W_1 may be reflected back through the optical system on to the photometer prism *P* by another right-angled prism, *TC*. The photometer prism is so constructed that the two beams, from W_1 and W_2 respectively, are brought together into the neighbouring parts of a simple bipartite field, separated by a sharp dividing line. The aperture *K* limits the field of view and is of such a size that it subtends approximately 2° at the eye, which is placed at the focus *E* of the objective *O*. An exit pupil, of diameter = 0·040 in., is placed at *E* to limit the width of the spectrum entering the eye. To control the intensities of the primary beams, neutral wedges *H*, the positions of which are controlled by the observer, are mounted in front of the prisms *R*, *G* and *B*. It should be pointed out that neutralization of the dispersion of the beams is effected on their return through the system so that two parallel beams, one from W_1 and one from W_2, fall on the prism *D*. The optical system also is so arranged that the returning light traverses a lower part of the system than that traversed by the incident light. This diminishes to a large extent troubles caused by back reflections from the various glass-air surfaces, and renders the apparatus effectively free from stray light. A fuller description will be found in a paper by W. D. Wright[1].

Owing to the existence of a small circular stop at *E*, instead of a slit, the light entering the eye is not strictly monochromatic. The integral effect of the light is to produce a colour sensation that can be matched with a monochromatic radiation, and when the apparatus is used as a colorimeter this arrangement is entirely satisfactory. But for a reliable determination of the hue-discrimination curve it is essential that the variable half of the field should be illuminated with a beam as nearly monochromatic as possible. This condition was obtained by placing a narrow slit in front of the prism *TC* in spectrum W_1 in such a position that a slit-image was formed in the centre of the exit pupil[1]. During the experiment the width of the slit was adjusted so that at no part of the spectrum was the wave-length band greater than 0·0010μ., while at the blue end it was considerably smaller. The only disadvantage produced by this modification was the slightly streaky appearance of one half of the field. This effect, however, was very slight and insufficient to cause any discomfort or difficulty to the observer.

In the method adopted for measuring the steps, the other half of the field was illuminated with light from various fixed points in the spectrum W_2 and used merely as a comparison colour. There was therefore no need to take special precautions to ensure strict monochromatism in this beam.

The only other modifications in the apparatus that had to be made were the provision of a scale in W_2 in order to set *R*, *G*, or *B* at the required points in the spectrum, and the addition of a mechanism to enable the observer to control the position of the prism *TC* in the W_1 spectrum. A simple device consisting of a

drum, string, pulleys and counterbalancing weights provided a fine control, easy to operate and almost entirely free from backlash. The settings were, of course, read direct from the spectrum scale in W_1 and were quite independent of any backlash or variance in the mechanism.

Before any observations were made the observer was dark-adapted for about 15 minutes, and then the actual experiment was performed as follows. One half of the field was illuminated with light of a certain wave-length coming from spectrum W_2, while the other half consisted of light of the same wave-length but coming from W_1. The luminosities of the two colours were then adjusted to equality by means of the neutral wedge. When an accurate match had been obtained the observer very slowly moved the prism TC through the spectrum, at the same time keeping the luminosity equal for both halves of the field by means of the wedge control. A position for the prism was eventually found where the colours of the two halves of the field were just perceptibly different and the reading of the wave-length scale was recorded. The prism TC was then moved in the opposite direction until the colours of the two halves of the field were again just perceptibly different and the wave-length was again recorded. This process was repeated twice for each wave-length, and after a little practice it was found that readings could be repeated with a high degree of accuracy considering the nature of the observation being made.

If the repeat readings did not agree well, more were taken. The average for each reading was found and the difference between them represented two colour steps. Half this difference gave one colour step, but only approximately in view of the fact that the lengths of the steps vary continuously throughout the spectrum. The colour steps, or difference limina, were found at various wave-lengths, and curves showing this relationship were obtained. The curves were confirmed with a repeat series of observations, and in the case of the authors the observations were repeated three or four times.

§ 3. RESULTS

It has been suggested in some quarters that for the results to be of full significance the determinations should be made at a uniform intensity-level throughout the spectrum. Laurens and Hamilton[2], however, have shown that the size of step is practically independent of intensity over quite a wide range, and this point was investigated and confirmed by the authors, although it was found that when the intensity was greatly diminished, say to about 0·01 of its original value, the colour limen did increase appreciably. As a uniform intensity-level would have involved a very low intensity throughout the experiment, it was considered better to allow some variation, but between wave-lengths 0·65 μ. and 0·48 μ., the range was only 4 to 1, with an average retinal illumination of the order of 70 photons, while the total range to the blue end of the spectrum was about 20 to 1.

Considerable scattering of the points was obtained in the initial observations, in consequence of lack of familiarity with the apparatus and the type of observation, but reasonably consistent and reproducible results were obtained after practice.

A smooth curve had, of course, to be drawn through the points, but with rare exceptions no point lay off the smooth curve by more than 10 per cent of the value of $\delta\lambda$ at that point.

In figure 2 the smooth curves for five observers are given, and a mean of their results is shown in figure 3. The similarity in general characteristics of all five curves. is very striking. The magnitude of the steps naturally varies from observer to observer, but each curve shows a broad minimum at about $0{\cdot}60\,\mu$. and a somewhat sharper minimum at about $0{\cdot}49\,\mu$. With one exception each curve has a secondary dip at the violet end of the spectrum between $0{\cdot}45\,\mu$. and $0{\cdot}46\,\mu$., but what is more

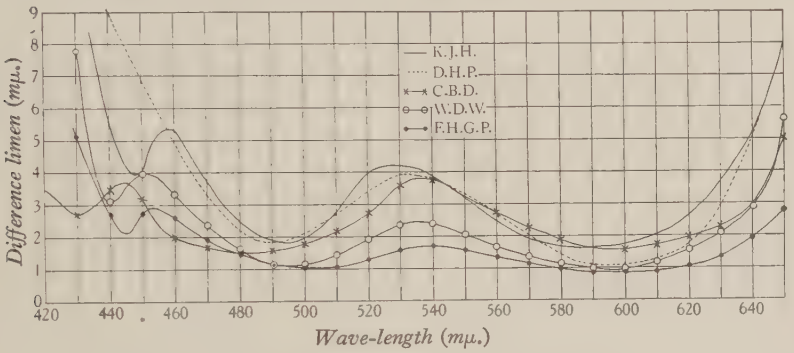

Figure 2. Hue-discrimination curves for five normal observers.

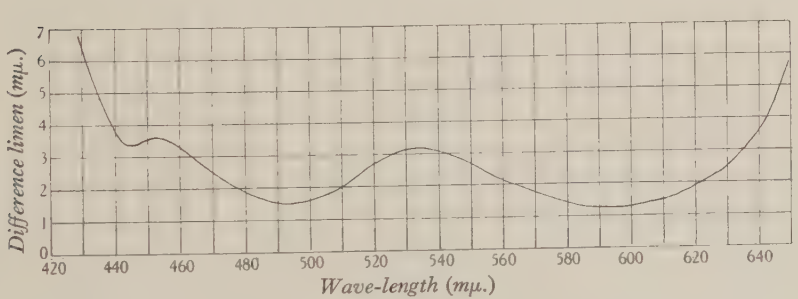

Figure 3. Mean hue-discrimination curve for five normal observers.

important, in relation to previous investigations, is that there is no secondary dip whatever in the red. In the next section it will be shown that a dip of this sort at about $0{\cdot}63\,\mu$. has been found in a number of previous investigations, and its absence is the most striking difference between these results and those of earlier investigators. As it was important to ensure that observational errors were not masking the dip, that region of the spectrum was checked with particular care. The scattering of the points was probably greater here than at any other point in the spectrum, possibly on account of the greater effect of colour-adaptation in the red, but no indication of a dip was obtained.

Some doubt has also been raised as to the existence of the dip in the violet, and it has been mentioned above that for one observer no dip was found. This result

may arise from the decreased sensitivity, due to the lower intensity of the field, as a result of which the steps may have been large enough to mask any dip that might exist. On the other hand, luminosity-measurements made on this observer showed that his blue luminosity was not appreciably lower than for the normal observer.

The curves as a whole have the characteristics that might be anticipated from a cursory glance at the spectrum. The part of the spectrum where a minimum step exists must obviously occur where there is a rapid change in colour; thus in the yellow where the colour turns redder on one side and greener on the other, in the blue-green where it turns bluer on one side and greener on the other, and in the violet where it becomes redder or bluer, minimum steps would be expected. But between 0·61 μ. and 0·65 μ. the colour changes steadily to a deeper and deeper red, and the existence of even a secondary minimum would be most unexpected. A rough analysis of this sort could not be used to discredit direct observations if they indicated a minimum, but it can be used to show that the curves now obtained have a good deal of logic to support them.

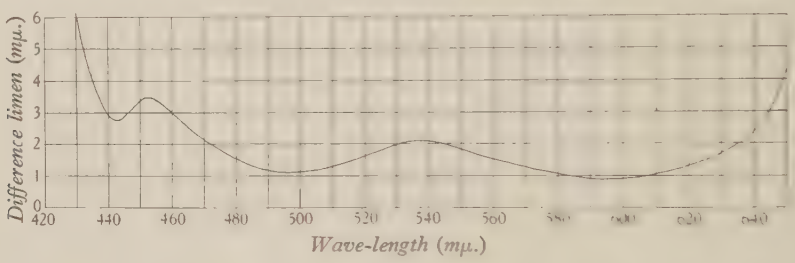

Figure 4. Mean of authors' hue-discrimination curves.

The process of averaging the curves inevitably tends to flatten out the dip in the violet region, and for this reason an average of the authors' curves alone has been made and is given in figure 4. This curve is perhaps more representative of the normal eye than figure 3, for if we imagined a very large number of curves to be available the dip in the violet would probably be averaged out altogether. Hence an average curve may not necessarily represent the visual characteristics of a normal eye. The data for the authors' eyes were obtained after more practice and were repeated more often than in the other cases, and as the two curves were remarkably similar, the mean curve of figure 4 is perhaps more valuable than figure 3. Although the point is of no special theoretical significance, it is of interest to note that the data in figure 4 imply that the authors can distinguish 150 different hues in the spectrum.

It should be mentioned that a preliminary test on the colour-mixing characteristics of the observers was made and none of those for whom curves are given were abnormal. A sixth observer was found to be anomalous, but although his discrimination curve was taken publication will be deferred until a more comprehensive study of anomalous vision has been made.

§ 4. PREVIOUS INVESTIGATIONS

The chief experimental investigations during the past 50 years have been those carried out by König and Dieterici[3], Steindler[4], Jones[5], and Laurens and Hamilton[2]. Valuable theoretical discussions have been made by Helmholtz[7] and Judd[9].

König and Dieterici[3] tested their own eyes by making the two halves of the field similar instead of just noticeably different. By repeating the setting 50 times the mean error of adjustment gave a measure of the sensitivity for the particular wave-length tested, and similar experiments repeated throughout the spectrum enabled the complete hue-discrimination curve to be drawn. The apparatus and conditions of observation appear to have been satisfactory. The two curves thus obtained are shown in figure 5. It should be noted that this method of measuring the sensitivity must give appreciably smaller values for $\delta\lambda$ than those obtained by the just-noticeable-difference method. It will be seen that no dip in the red was found.

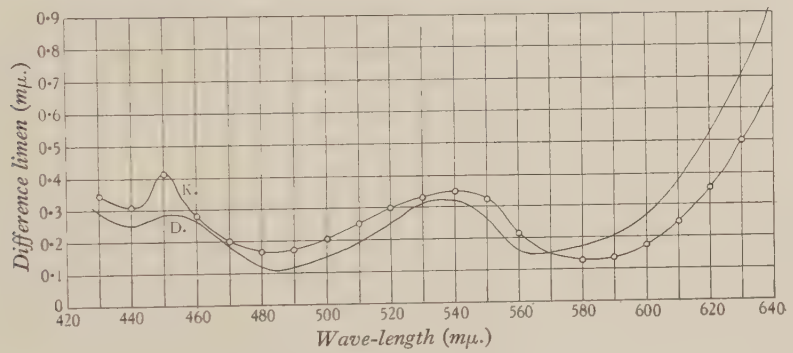

Figure 5. Hue-discrimination curves of König and Dieterici[3].

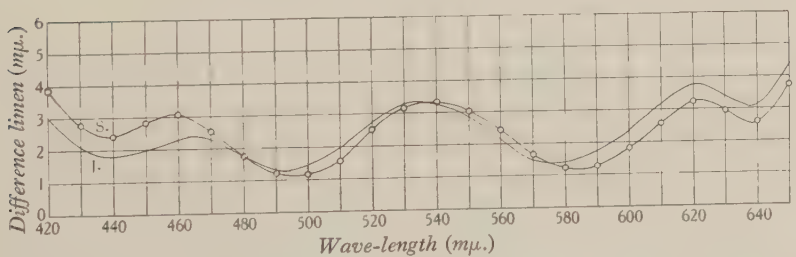

Figure 6. Hue-discrimination curves of Steindler[4] and Jones[5].

Steindler[4], using a concave grating, obtained a first-order spectrum 85 cm. long. She made, on 12 normal observers, measurements of the minimum change of wave-length perceptible as a difference of hue between two juxtaposed fields. The precaution of equating the luminosity of both halves of the field was not generally adopted, and the field of view was apparently of a rather low intensity. The value of her results seems rather doubtful, especially as she obtained a very extraordinary curve for the hue-discrimination of colour-blind (protanopic) observers. Her mean curve for the normal observers is given in figure 6, curve *S*.

Jones[5] investigated the hue-discrimination of only one eye, his own, but the observations appear to have been made under proper observational conditions. Some instrumental difficulties are recorded and although these are reported to have been overcome, the work was not extended to other observers as originally intended. The curve he obtained is also shown in figure 6, curve *J*; the close agreement with Steindler's curve is somewhat unexpected, especially as the secondary dip in the red occurs at a longer wave-length than in any of Laurens and Hamilton's results. Both Steindler and Jones give secondary minima in the violet.

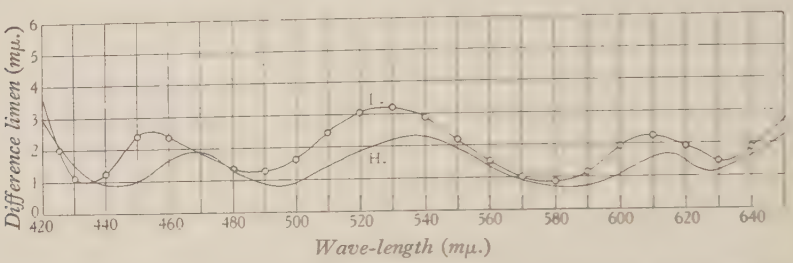

Figure 7. Hue-discrimination curves, each of which is the average of 16 complete series: Laurens and Hamilton.

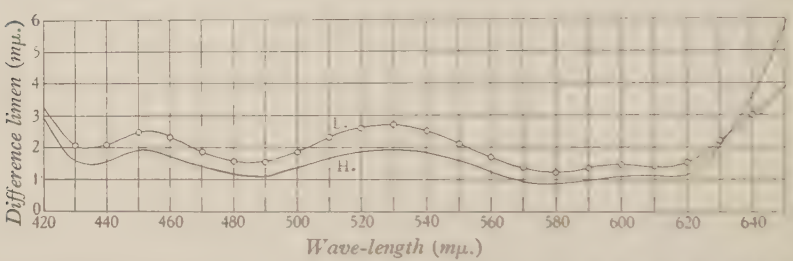

Figure 8. Hue-discrimination curves based on the simple averages of the values of the difference threshold at every 10 mμ. 16 tests on both Laurens and Hamilton.

Laurens and Hamilton[2] obtained results for 10 normal observers. Their first apparatus, which was used to obtain data for 7 subjects, was subsequently discarded in view of certain disadvantages. Four subjects, two of whom were Laurens and Hamilton, were tested with an improved apparatus, but the experiments cannot be regarded as very satisfactory since it was necessary for an assistant to make the changes in wave-length. The observer was required to state whether he observed a colour-difference or not as the sizes of the steps were increased. The point at which a colour-difference was noticed was recorded and the sizes of the steps were deduced from these readings. Anyone familiar with limen-measurements will realize the unreliability of the results made on this basis, unless a large number of answers are obtained and the results subjected to the proper analysis of observations. Moreover, only a very low intensity-level (4 photons) was used. These conclusions are confirmed by the considerable variations shown even among the curves for the same observers. Thus in figures 7 and 8 two sets of data for the

same observers, Laurens and Hamilton, both taken on the improved apparatus, are reproduced. In the region that is most in doubt, the red, a marked difference between the two diagrams is seen. Evidently no evidence for or against the existence of a secondary dip can be deduced from these results.

Tyndall[6] has recently published a hue-discrimination curve for his own eye that shows a very small dip in the red. The dip, however, is so small that the observations hardly justify its existence.

Helmholtz[7] carried out a theoretical investigation of the relation between König's fundamental sensation curves and the hue-discrimination curve, and deduced a result in remarkably good agreement with König's discrimination curve given above, which shows no dip in the red. The basis of Helmholtz's calculations was criticized by Schrödinger[8], but the curve obtained by the latter was substantially the same.

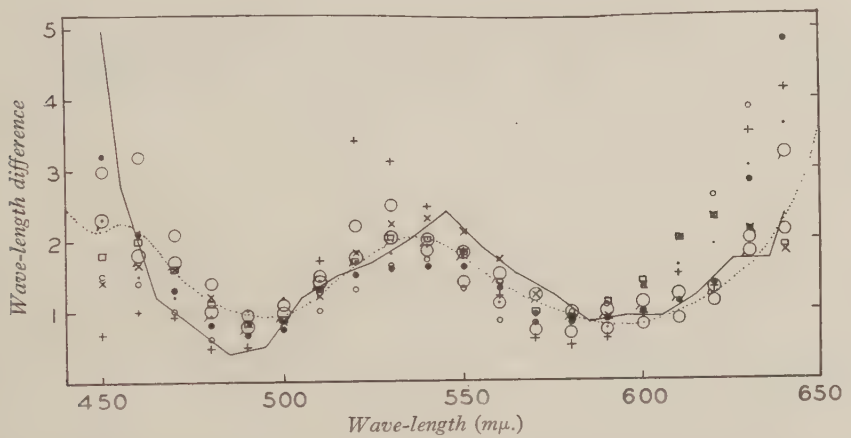

Figure 9. Experimental results from various observers as given by Judd. Continuous line represents Judd's theoretical curve; dotted line represents curve shown in figure 3. · Koenig[1]; ○ Dieterici; ● Koenig[2]; + Exner; □ Steindler; ○ Tyndall; ○ Uhthoff; × Jones; — from empirical law.

Judd[9] has analysed some of the experimental data on hue-discrimination and reproduced them in one diagram, which is also shown here in figure 9. He has developed a theoretical curve to fit the points approximately; this is shown by the continuous line in figure 9. The dotted curve is the mean curve of figure 3 of the present paper, superposed by the authors to show the extent of the agreement. This is very satisfactory, although it must be admitted that averaging a number of different investigations will always tend to smooth out any secondary minima that may exist.

Houstoun[10] discussed the need for a change in colour-quality in the red if a secondary minimum really existed. It has been suggested that the red might become slightly blue, but modern colorimetric measurements have shown conclusively that this does not occur.

§ 5. CONCLUSIONS

How much weight to attach to various investigations is difficult to decide, but in this case the difficulty could be avoided by ascribing the disagreement of the results to the variation in colour-vision of the observers. Yet the authors feel that this is not the correct answer to the question. They believe that, while variations in detail must be expected from observer to observer, the general characteristics of hue-discrimination do not vary in the manner required to include, under this head, all the variations that have been reported.

The review of previous work has shown that there are at least as many data in agreement with the curves given in this paper as in disagreement. Yet for some reason the curves usually quoted in text-books and elsewhere have almost invariably been those having the secondary dip in the red.

At present little can be said beyond an expression of hope that it will be possible in the near future for some other investigator, with different apparatus but using the proper conditions and precautions, to repeat the experiment with a number of normal observers. It is anticipated with confidence that the results obtained here will be generally confirmed.

§ 6. ACKNOWLEDGMENTS

In conclusion the authors desire to express their very sincere thanks to those who spent considerable time and trouble in making observations, to Prof. L. C. Martin, and to the Medical Research Council for their financial assistance and for permission to publish these results.

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DISCUSSION

Mr C. R. Darling. The late J. W. Lovibond informed me many years ago that in using his tintometer to match different parts of the red end of the spectrum, he found one part which presented more difficulty than the portions on either side. He was of opinion that this was due to the human eye being less sensitive to this particular band than to the remainder. The conclusions reached in the present paper do not confirm this view, but as the question is of practical importance it would seem desirable to make a special investigation from the standpoint of colour-matching.

Prof. L. C. Martin. I think the authors' results deserve careful attention, as experiments of this kind make very stringent demands on the perfection of the apparatus. In this case the freedom from stray light, and the good observing conditions generally, lend considerable weight to the conclusions reached. It might perhaps be suggested that some attention should be directed in the paper to the influence of the intensity-drop at each end of the spectrum, which may, I suppose, be taken to explain the rapid increase in the limen at the ends of the curve. How far would a red maximum and minimum have to be displaced towards the end of the spectrum before being masked completely by the intensity effect? I quite agree with the idea that individual differences in the curve may be of very considerable interest in the discussion of theories of colour-vision and its variations.

Dr W. S. Stiles. The authors mention that their own hue-discrimination curves were determined on several occasions. It would be interesting to know the order of magnitude of the variations exhibited in these repeat observations, and whether any general change throughout the spectrum occurred which could be interpreted as a distortion of the curve or a bodily shift up or down. Marked day-to-day variations are frequently observed in brightness-difference threshold measurements and must be attributed to changes in the visual mechanism. If the three mechanisms postulated by the trichromatic theory are subject to such changes and are so in different degrees, distortion of the hue-discrimination curve may result. The authors are, I think, a little hard on the method of determining thresholds by answers given by the subject. Admittedly laborious, the method is fundamentally sound and can be employed under conditions where setting by the subject is out of the question. No criticism of the use of setting by the subject in the present investigation is intended.

Mr J. Guild. In this paper the authors present us not only with new observations on wave-length discrimination by the eye in virtue of the accompanying change of hue, but also with a very useful digest of previous results. The importance of fresh observations in these discrimination problems cannot be over-emphasized. The determinations are of a very difficult kind and peculiarly liable to systematic

errors, and the more independent determinations we have the less likely it is that the same systematic errors will pervade them all.

The authors quite correctly perceive that in matters of this kind it is much more important to obtain thoroughly reliable results for one reasonably normal observer than to devote the same time to obtaining the average results of a cursory examination of a large number of observers. The statistical bogy plays too large a part in the general attitude to problems of this kind. Where the object of an investigation is to set up some convenient social standard, as, for instance, a "standard observer" for the purpose of giving definiteness to technically important photometric and colorimetric quantities, statistical data representing a hypothetical "average" or "normal" individual are of importance; but where the object of an investigation is to establish certain characteristics which are inherent in the nature of the human machine, it is much better to devote the available time to the careful examination of one individual, or at most two or three, than to spend the same time rushing a large number of unskilled subjects through tests of a necessarily hurried character.

The authors also point out, what is generally overlooked, that average results may in fact minimize or conceal characteristics which are found in every individual.

In liminal experiments the question of tolerances is of the utmost importance. In fact the experiments are *prima facie* determinations of tolerances in some specified variable. If this were the only experimental variable entering into the experiments everything would be straightforward, but it is not; the brightness varies continuously along the spectrum provided by the instrument, and has to be adjusted to equality at each observation. In theory the observer, having made this adjustment, has nothing to detect except difference in hue. In practice the equalization of brightness is only approximate: there is a practical tolerance in the intensity-adjustment as well as in the hue-adjustment. When dealing with liminal differences it is impossible to distinguish completely between differences in hue and differences in brightness. The observer, having done the best he can to equalize the brightness, assumes that any difference he sees is difference in hue, but if the brightnesses are not in fact equal, the difference he sees may be compounded of a difference in brightness and a difference in hue, each of which by itself may be subliminal, though together they produce an appreciable change in sensation.

We cannot assume on *a priori* grounds that the effects of the brightness-tolerance, even if non-systematic, will be eliminated in the mean of a large series of observations, but we can be sure that they will not be eliminated if there is any predisposing cause of systematic error in the brightness-adjustments, and it is beyond question that in all colour-matching apparatus there are factors which predispose the various tolerances to be more on one side of the "true" adjustment than the other. To discuss these factors would occupy too much space, but they may easily be responsible for the fact that some investigators obtain results exhibiting features which are absent from those obtained by others, and we are faced also with the more disquieting possibility that since all investigations of this kind have had much

in common as regards general experimental method they may be affected in common by certain systematic errors.

With regard to the most important difference between the present results and those hitherto regarded as most reliable, namely the absence of a minimum in the red, this is certainly more in accordance with my own experience obtained in other ways.

I also agree with the authors that there is no theoretical reason so far known for expecting any such minimum; though too much stress must not be laid on this, as the theory of the relation of the wave-length limen to the other quantitative facts of vision is still, in my opinion, on a very tentative and insecure basis. However, such theoretical guidance as we yet have indicates that there should be no secondary minimum in the red. Thus in so far as the present results differ from others I think they are better, but I cannot feel quite so certain that in every respect in which they all agree they truly represent exactly what they profess to represent.

I think that on several grounds the best way to make liminal determinations is not to determine the limen directly but to deduce it from the spread of a large series of attempts at exact matching. This eliminates a number of psychological elements which enter into the direct method, though it is still subject to possible systematic errors arising from instrumental factors.

As far as I can see the only way to eliminate these, by either the direct or indirect method, is to accumulate results obtained with apparatus differing as much as possible in its mode of operation. In particular, I think it might make an appreciable difference if matters were so arranged that the range of the spectrum involved in a set of observations were of constant brightness throughout, so that the operation of the wave-length drum produces change of hue only and no accompanying change of brightness to be remedied by the auxiliary brightness-control. This would be difficult but should not be impracticable.

I hope that Dr Wright and his colleagues will continue to bring their experimental skill to bear on these visual problems, for the work is very much needed.

Mr T. Smith. The authors have referred to the dangers of combining results reached with different eyes, but this only increases the importance of making the fullest use of information obtained for the same eyes. I do not know whether Dr Wright has determined the visibility function for his own eyes, but he has certainly carried out much work on colour-matching. I should like therefore to ask him whether he has been able to infer from his own results in this and previous work any conclusions bearing on hitherto doubtful points in the theory of vision other than those mentioned in this paper, which relate to his last investigation alone.

Authors' reply. In reply to Mr Darling: We should suspect that the difficulty experienced by Mr Lovibond in matching a certain part of the red end of the spectrum with his tintometer was due to the nature of the colour-filters used rather than to any reduced sensitivity of the eye in that part of the spectrum. The problem really concerns the colour-mixing characteristics of the eye, and no results

on colour-mixture obtained in recent years have suggested that a band in the red end of the spectrum is harder to match than the radiations on either side. The only bearing that hue-discrimination data have is as an indication of the limits within which two colours will still appear to be matched. If Mr Lovibond's idea that the eye is less sensitive to the particular band were correct, then actually that colour would be easier rather than more difficult to match, because the range over which a match would appear to hold would be greater.

Prof. Martin draws attention to the effect that the low intensity at each end of the spectrum might have in producing large hue-discrimination steps. It is improbable that any action of this sort is at work at the red end, since, at a wave-length of $0{\cdot}65\mu$., the steps have already become large although the intensity is still high. A red maximum and minimum could hardly be masked by an intensity effect, since the steps would become large as a result of the small hue-discrimination before they became large as a result of low intensity. At the blue end of the curve there is little doubt that some of the increase in the size of step is due to the low intensity.

The query raised by Dr Stiles regarding the manner in which our hue-discrimination curves varied on different occasions cannot be answered very conclusively. One of us (F. H. G. P.) found little difficulty in repeating his curve to a surprisingly close degree on different occasions, but the other found that his sensitivity as a whole was liable to vary, producing a general increase or decrease in the size of his steps right through the spectrum. No distortion of the curve was found other than that due to accidental variations, although the possibility that changes in the visual mechanism may produce changes in the discrimination curve cannot be ignored. No evidence of such an effect was obtained.

We did not intend to convey an impression of distrust towards the method of determining thresholds by answers; our criticism was directed towards the manner in which the test appears to have been carried out by Laurens and Hamilton and not to the test in itself. The method is, we believe, perfectly sound for the usual type of threshold observation if correctly conducted. Nevertheless, in this case, in view of the simultaneous action of intensity and colour limina, the method of answers calls for very careful application and analysis if the results are to mean anything. The discussion below is particularly relevant to this point.

It is very gratifying to find that Mr Guild is in agreement with so many of the points we have made and to note that from his own experience he would not anticipate any dip in the red. We do not, however, share Mr Guild's doubt as to the quantities really represented by the hue-discrimination curve. His analysis of a colour limen into a subliminal intensity-difference and a subliminal hue-difference is correct for what we may call a static observation, that is an observation in which we merely view two fields and decide whether a difference exists or not, without having the power to control either the intensity or the wave-length of the two patches. But in the method of observation we adopted, the wave-length of one half of the field was altered and then the intensity of the other half was adjusted to see whether any difference that initially existed disappeared. The observer was only satisfied in his setting when he arrived at a wave-length step that produced a just-noticeable difference when the intensities of the two halves were exactly the same.

Although it might be impossible to say when this equality held, yet since the intensity of one half could be made greater or less than that of the other, then at some intermediate position they must be equal. This state was observed with all the other states, and if the colour-difference had been subliminal, no difference would have been seen at this point and the observer would have increased his step until a just-noticeable difference was perceived. The value finally recorded is therefore a colour limen alone and not a combination of intensity and colour limina.

On the other hand, this would not apply in the method of matching preferred by Mr Guild. In this method the observer would have to alter the intensity and wave-length until the two halves appeared alike, which would imply that a subliminal intensity-difference plus a subliminal colour-difference were together still below the threshold. It would be impossible to separate one from the other, and the final distribution of the points would be a complex function of the two limina. Thus this method is actually more and not less uncertain than the one we have used. An instrument in which the intensity was automatically adjusted to a constant level would no doubt be of value, but even there the two halves of the field would have, initially, to be brought to the same intensity by eye, and a subliminal error in this setting would produce a systematic error of the very type that we wish to avoid.

We conclude from this that the direct determination of the colour-step is, after all, the best method to adopt and the results so obtained do in fact represent the quantities that they purport to represent. This does not mean that we can immediately relate the discrimination curve to the visual mechanism; that will undoubtedly be a difficult process, and to illustrate the difficulty we need cite only one factor, namely adaptation, that must play an important part in determining the shape of the hue-discrimination curve.

In reply to Mr Smith's query regarding doubtful points in the theory of vision, both of us have determined our luminosity curves, trichromatic coefficients of the spectral colours and, of course, our hue-discrimination curves. These results do not differ seriously from previous determinations of the same functions except at the points mentioned in this paper. Any theory of colour-vision must account for these fundamental experimental results, but they do not provide new information of the type that assists in the evolution of a more satisfactory theory. Work on adaptation recently published* has, however, led to some significant conclusions and it would seem that more complete information on the various characteristics of colour-blindness provide the other most hopeful avenue of attack. Work on these lines is in hand, but until the results are completed we should prefer not to discuss theories of vision in detail.

As a complete set of curves may be of interest to others working in this field, we hope to publish our luminosity, colour-mixture and hue-discrimination curves shortly with, if possible, the addition of some measurements on saturation-discrimination.

* *Proc. R. S.* A, April, 1934.

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A HIGH-SPEED MOTION-PICTURE TIMING-SYSTEM

by E. A. SABINE, Technical Consulting Department,
Western Electric Company Ltd

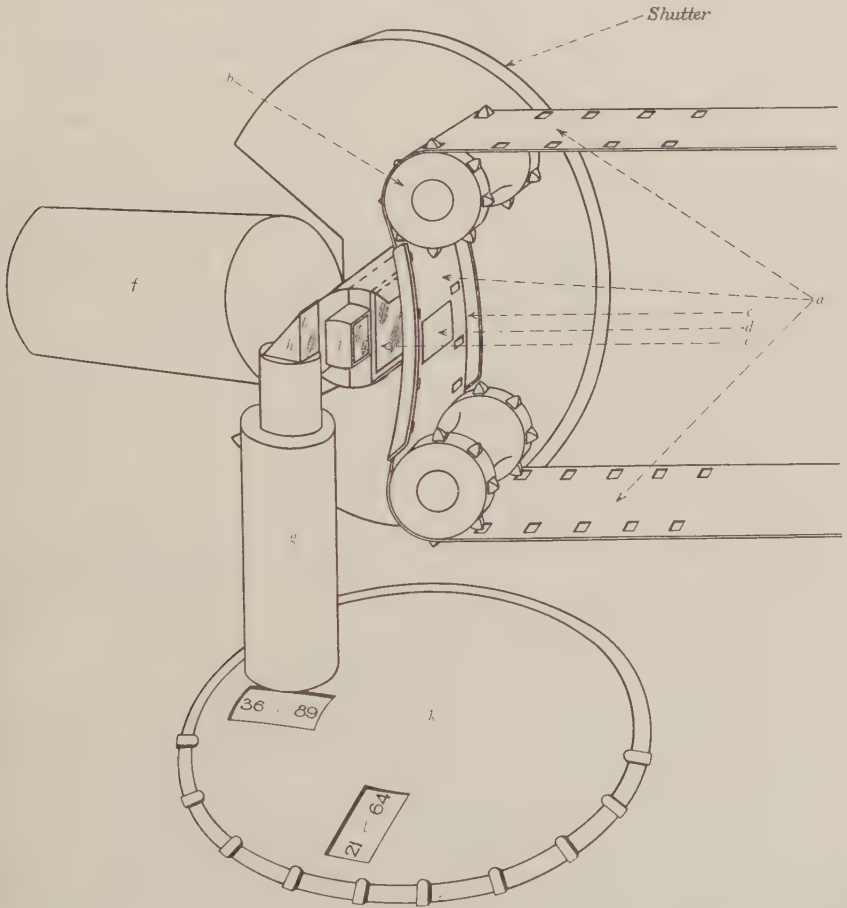
Demonstration given on March 2, 1934

§ 1. GENERAL DESCRIPTION

THE high-speed timing-system to be described was developed by the Bell Telephone Laboratories in conjunction with the Eastman Kodak Company and consists fundamentally of three main elements: a high-speed motion-picture camera, a high-precision electric clock, and a frequency-generator employing a tuning-fork to control the alternating current actuating the clock mechanism. The camera, which uses standard 16-mm. reversal safety film, has two lens systems so arranged that, when the camera is used to photograph any occurrence, the moving dials of the electric clock are also simultaneously photographed upon each frame of film beside the picture. Upon viewing or projecting the image so made, the time at which any event occurred can thus be read immediately from the direct time-record given with the picture of the occurrence.

The camera is available in two models, fundamentally similar but differing as regards the driving-motor and various secondary features, one being a high-speed model taking up to 250 frames per second and the other an ultra-high-speed model taking from 300 to 2000 frames per second. Owing to the excessive noise, vibration and wear, and the expensive construction and upkeep which would be inherent in a camera of conventional intermittent design if run at high speeds, such mechanism would be highly undesirable for this purpose, and the cameras are, therefore, of the continuous-movement type wherein all moving parts operate at constant speed. The film *a* tensioned by means of a spring-mounted top sprocket *b* is pulled continuously across a gate *c* having an aperture *d* slightly larger than the film-frame size in the vertical dimension. The projection of the image on to the moving film is accomplished by means of a uniformly-rotating plane-parallel plate of glass *e* which is interposed between the lens *f* and the film and acts as a plane prism. As the prism revolves the effect is to move the image in the film-plane downwards at the same speed as the film during the instant for which the shutter is open, thereby enabling a sharp picture to be obtained. Two pictures are thus made at each revolution of the plane prism, and every frame on the film is exposed. Framing is accomplished either by using an external mask or by means of small uniformly-rotating blades located respectively at the top and bottom of the picture aperture. The edges of the blades follow the frame lines of the picture down as the film moves. An auxiliary lens *g* and prism system *h* projects an image of the timing dials *i*,

suitably illuminated, through the rotating prism *j* so that this image is registered on the side of the film beside the picture in a manner similar to that in which the sound track is registered on sound films, except that in the present case the image of the timing dials is complete and appertains to the action photographed upon the frame beside which it appears, whereas sound track is of a continuous nature and the registration of sound applying to any particular frame is displaced approximately $14\frac{1}{2}$ inches in advance of the frame.



The high-speed or type-I camera is equipped with a lens-turret mounting, a $1\frac{1}{2}$-in. F 2 Kodak anastigmat lens and a 4-in. F 4·5 Kodak telephoto lens and the ultra-high-speed, or type-II, camera is fitted with a single $1\frac{1}{2}$-in. F 2 Kodak anastigmat lens; other lenses can readily be fitted if desired.

Speed-control of the type-I camera is effected by an electric governor, the setting of which can be varied by means of a speed-control knob at the rear of the motor. In the case of the type-II camera the speed is controlled by means of a rheostat in association with a series motor which for short periods can be considerably overvolted. Under such conditions the motor can be made to drive the camera

at such a speed that the motor itself runs at about 20,000 r.p.m. Internal gearing of the camera increases this speed up to 80,000 r.p.m. of the shaft which rotates the plane parallel prism. Since two images are registered on the film at each revolution of the plane prism, it follows that the linear speed of the film can be about 160,000 pictures per minute or approximately 2600 pictures per second.

In practice, excellent photographs have been secured at 2000 frames per second, and these when projected at normal speed give slow-motion pictures about a hundred times slower than the original action. This is thirty times slower than ordinary slow motion or fifteen times slower than the ultra-slow motion of the cinema.

§ 2. ELECTRIC-MOTOR CLOCK UNIT

The high-precision timing system which is an important feature of this apparatus consists of an electric clock which also is available in two types. The type-I clock unit, used in conjunction with the type-I camera, consists of an electric clock having three concentric dials reading minutes, seconds and hundredths of a second respectively to a total of 59 minutes, 59·99 seconds without repetition.

The type-II clock unit, used in conjunction with the type-II camera, consists of an electric clock having two concentric dials, one rotating once a minute and divided into 60 divisions for seconds and the other rotating once a second and graduated into 500 divisions each corresponding to 0·002 sec. Each division is further subdivided by a diagonal line which when read with reference to a stationary central vertical line permits time to be read down to 0·001 sec.

The dials are actuated by a specially developed salient-pole electric motor driven by alternating current, the frequency of which is regulated by a thermostatically controlled tuning fork calibrated to 200 c. sec. For short periods not exceeding 2 minutes the accuracy of the timing system with reference to the total time elapsed is within 0·05 sec. For longer intervals the precision is within 1 part in 25,000. In comparing closely adjoining pictures these limits do not, of course, apply, and it therefore becomes feasible to read to 0·001 sec. in such cases.

Two windows are provided for observing the position of the clock dials; they are cut in the resetting-plate *k* which is turned in a clockwise direction to reset the dials after each occurrence has been timed. One of the windows *l*, illuminated by two automobile head-lamp bulbs and small condensing-lenses, is located directly under the auxiliary lens-and-prism system, and reveals the dials that record absolute time. It is the image of this window which is photographed upon the film. Since, however, this window is not visible to the operator when the camera is operating, a further window *l'* is provided at an angle of 90° to the left of the objective window. The reading as observed in this window will, of course, be 15 minutes, 15 seconds and 25 hundredths in advance of the absolute time, except when the dials are in the reset position prior to starting. In this position the hundredth second dial is advanced 0·007 sec. in order to compensate for the time required for the operation of the magnetic starting clutch, and the time which is then indicated by the dials in the observation window is 15 minutes, 15 seconds and 25·7 hundredths.

The clock may be started or stopped at the camera location by means of two momentary-control keys located on the clock base, so arranged as to discharge polarized condensers in opposite directions and thus actuate relays which engage pawl-and-ratchet clutch mechanisms which, in turn, positively start or stop the clock dials. Alternatively the timing system may be started by means of a switch from some location remote from the camera.

§ 3. THE FREQUENCY-GENERATOR

The frequency-generator unit is a portable unit containing an automatically temperature-controlled tuning-fork and an amplifier to increase the output-level to a suitable value for the operation of at least two motor clock units. The primary purpose of this unit is to provide a sufficiently precise basic time rate for the system, and a secondary service is to provide operating current for the circuit.

The tuning fork vibrates at 200 c./sec., the vibration being started and sustained by the interaction of magnetic coils. Its temperature is controlled by electric heater elements which are in turn controlled by a mercury column thermostat embedded in the metal of the fork. The fork assembly is sealed in a heavily heat-insulated metal enclosure. This heat-regulation is such as to permit the fork to maintain the required precision over a temperature-range of $-20°$ to $+120°$ F. The primary requirement of the fork is temperature stability and it is, therefore, essential that the fork be connected and allowed to reach the stable temperature for at least one hour before time observations are made. The fork will maintain a precision of 1 in 25,000 over the previously mentioned temperature-range, with the commercial variations in the gain and characteristic of the amplifier, if the specified requirements as to operating-voltage adjustment and circuit-operation checking are correctly observed.

§ 4. CONCLUSION

In conclusion, it will be evident that the performance of this timing-system is comparable with relatively high-precision methods of laboratory practice, and will provide a commercial tool capable of laboratory precision.

To illustrate the performance of the equipment, some films will be shown which were taken for Dr Allan Ferguson in connexion with his discourse on surface tension before the Royal Institution. One of the subjects under observation is the splash of a drop under varying conditions, and it is interesting to recall that Worthington in the latter part of the nineteenth century occupied himself for several years in obtaining, with the aid of a highly ingenious electro-photographic device, a series of instantaneous photographs illustrating the various phenomena. He produced some hundreds of excellent photographs, but since these are simply a selection of stills the interpretation of continuity is largely a function of the imagination of the observer.

The films taken of the same phenomena for Dr Ferguson were taken without any special preparations and the complete series was shot in three afternoon sessions of 2 hours each.

CONDENSED TABLES FOR COLOUR COMPUTATION

By T. SMITH, M.A., F.Inst.P., F.R.S.*

Addendum to discussion

Mr D. B. Judd (Bureau of Standards, Washington, U.S.A.). I have two objections to the derivation of condensed tables according to the author's method. The first is that high-order Lagrangian interpolation is from a practical standpoint an unsuitable base from which to start the argument; the second is that it seems not to have been proved that the purely theoretical problem can be "solved by the construction of special tables."

In order to explain my first objection I refer to the second paragraph of the paper: "The formally correct procedure is then to measure at the 10-mμ. intervals, to interpolate values for the 5-mμ. intervals by a suitable formula, and then to compute exactly as though observations had been made at the 5-mμ. intervals." This would be correct if it had specified a suitable *method* instead of a suitable *formula*. I am not at all sure that there is any formula generally suited to interpolation of spectrophotometric data; and, in particular, I am very sure that Lagrangian interpolation of the 20th order is not suited to all cases.

As an example, the accompanying diagram shows the spectral transmission of Signal Red Glass No. 126†. The circles show the measured transmissions and the dots indicate values obtained from Lagrangian interpolation of the 20th order. The curve is drawn so as to pass through both circles and dots; hence it represents closely the result of Lagrangian interpolation of the 20th order. It is seen that this curve is not satisfactory; it descends to impossible values of less than zero by amounts that are significant. The graphical method of interpolation, on the other hand, would quickly yield much less objectionable values, because values less than zero could easily be excluded.

The statement, then, that an abridged table gives the same result as would be obtained by using high-order interpolation for obtaining intermediate values of spectral transmission is not always an argument in favour of its adoption; in some cases it is an argument to the contrary.

Before explaining my second objection I ought to state that I do not know whether the author's tables really fail to give 10-mμ. sums in agreement with the official 5-mμ. sums referred to transmission interpolated by a high-order Lagrangian formula; but I doubt if they do because the proof given is not complete. By writing out the expression (p. 375) completely it is seen that

$$\Sigma bc = \Sigma b_0' c_0,$$

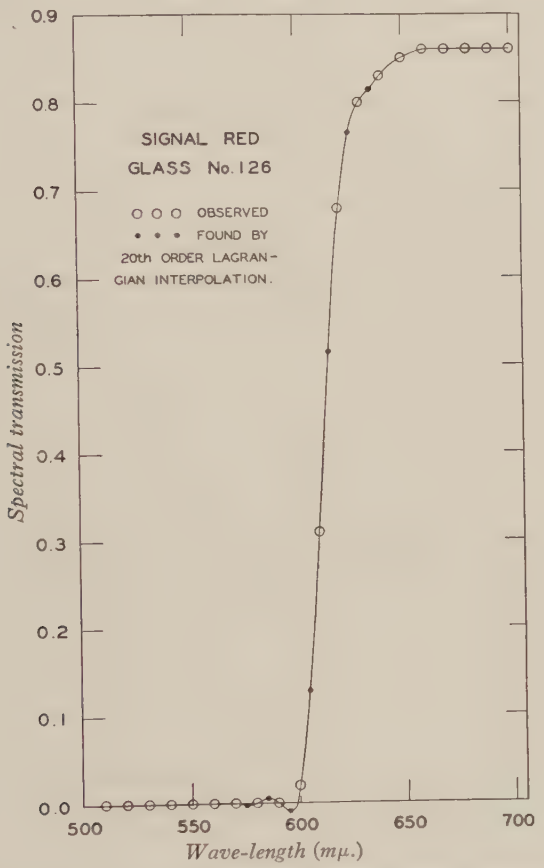
only provided

$$\Sigma b_e (\Sigma w_0 c_0) = \Sigma c_0 (\Sigma b_e w_0).$$

* *Proc. Phys. Soc.* **46**, 372 (1934). † *Proc. A. R. A. Signal Section*, **30**, 407 (1933).

But this is not the case; the final sum is not independent of the order in which the operations are carried out.

This concludes my objections, but I have a final comment to make. If condensed tables are to differ at all from alternate entries of the official tables* my impression would be that an average value of the function for the interval should be given instead of the value at the middle of the interval. But the interesting result obtained by the author in his note on integrals of products of experimentally determined



An example of 20th-order Lagrangian interpolation of actual spectral-transmission data. It may be noted that the interpolation by formula is not admissible because it gives values less than zero.

magnitudes† goes far toward persuading me that the gain to be had, if any, is not worth the trouble. Our experience agrees perfectly with that at the National Physical Laboratory described by the author's statement: "In fact, the standard-observer properties are smooth enough for the outstanding irregularities to produce no difference that could be appreciated in any tests that have so far been made...."

AUTHOR's reply. I should like to thank Mr Judd for sending a contribution to the discussion of my two papers. The points which he describes as objections are of

* As in *J. Opt. Soc. Am.* **23**, 365 (1933).

† Page 365 of this volume.

interest, and I am glad to have the opportunity of saying a little about them. I have evidently given Mr Judd the impression that I regard 20th-order Lagrangian interpolation as a generally applicable method of interpolating among spectrophotometric measurements. This impression is quite incorrect. The 20th order was found to be sufficient for accurate interpolation to four figures of the Standard Observer tables, when the condensed tables were being constructed. I have nowhere stated what expression is necessary or sufficient for the properties of a source or material. If the purpose of the interpolation formula were to give a value for the intensity of the radiation at a particular wave-length, it is obvious that a formula giving negative values would fail, or alternatively that the observed points in the neighbourhood considered were not dense enough for the formula selected to be satisfactory. But the presence of negative values of the kind illustrated does not mean that the use of this formula would probably give rise to any detectable error in the integrals we are considering. It will be observed that the false negative value tends to be balanced by excess positive values immediately on either side. Phenomena of this kind are familiar in mathematics and mathematical physics. For example there are well-known expressions for evaluating areas in which negative weights are given to some ordinates: and the applications of Fourier's theorem constitute an important group in which we obtain correct results in spite of features of the type to which Mr Judd has called attention. I am unable to agree with Mr Judd's argument that a feature of this kind is a valid reason for not using the formula.

Mr Judd's second objection relates to the basis on which the tables have been computed. He says in fact that the work rests on a false assumption. I admit that the basis of the work was explained very briefly, but the error is certainly in Mr Judd's remark, and not in my work. Stated more fully the foundation is

$$\Sigma b_r c_r = \Sigma b_{2r} c_{2r} + \Sigma b_{2r+1} c_{2r+1}$$

and

$$\Sigma b_{2r+1} c_{2r+1} = \Sigma b_{2r+1} \{w_1 (c_{2r} + c_{2r+2}) + w_3 (c_{2r-2} + c_{2r+4}) + w_5 (c_{2r-4} + c_{2r+6}) + \ldots\},$$

where the weights w_1, w_3, w_5, are independent of the value of r and depend only on the differences of the suffixes of b and c. In this sum we have only a finite number of non-vanishing terms, and so we do not change the value by regrouping the terms. We may collect together the terms involving c_{2r} and obtain

$$\Sigma b_{2r+1} c_{2r+1} = \Sigma c_{2r} \{w_1 (b_{2r-1} + b_{2r+1}) + w_3 (b_{2r-3} + b_{2r+3}) + w_5 (b_{2r-5} + b_{2r+5}) + \ldots\}$$

leading to

$$\Sigma b_r c_r = \Sigma b_{2r}' c_{2r},$$

where

$$b_{2r}' = b_{2r} + w_1 (b_{2r-1} + b_{2r+1}) + w_3 (b_{2r-3} + b_{2r+3}) + \ldots.$$

This is the result on which the calculations are based. It may be interpreted as showing what we ought to understand by Mr Judd's "average value of the function for the interval." It is this value which I have tabulated in the paper, and it is available for every one to use without any trouble at all.

I am glad to know that at the Bureau of Standards the Standard Observer tables have been found as satisfactory as regards smoothness as at the National Physical Laboratory.

REVIEWS OF BOOKS

Theoretical Physics, volume 2; *Electromagnetism and Optics*, by Prof. W. Wilson. Pp. xi + 315. (London: Methuen & Co.) 18*s*. net.

Prof. Wilson's first volume*, *Mechanics and Heat*, has already made many friends, and the volume now appearing is assured in advance of a warm welcome. More than half is devoted to classical electromagnetic theory. This is followed by two valuable chapters on electron theory, a short chapter on geometrical optics, and just over fifty pages on some major topics in physical optics which have not already been dealt with as branches of electromagnetic theory. The volume concludes with a chapter on "General questions concerning the propagation of light," leading up to the discussion of relativity, which, with quantum dynamics, is to be the subject of the third volume.

The skill of the writer and the largely mathematical form of his argument have made it possible to compress into a remarkably small space a surprisingly comprehensive survey of classical and immediately pre-quantum (the use of this adjective seems unfortunately unavoidable) physics. This compression has not, however, been achieved at the expense of clarity of exposition; in fact, several points which usually present difficulties to students are here discussed much more fully than is usual in treatises of this scope. The problem of the variety of electrical units in common use, which is often, for beginners, elevated to the rank of a quite unnecessarily imposing bugaboo, is here handled in a thoroughly competent and sensible manner.

While the selection of the subject-matter has necessarily been along conventional lines, the treatment is both markedly individual and consistent. A particularly valuable feature is that, while presenting the classical physics, the author has his eye continually, if unobtrusively, focused upon the later developments of the subject; again and again, details of the older theory which are of special importance in more recent work are given special prominence. The reader who uses this book intelligently can scarcely fail to see "physical theory as a coherent logical unity." The author has earned the warmest thanks of all teachers and senior students of physics. H. R. R.

Les Bases expérimentales immédiates de la Théorie des Quanta, by P. Auger, E. Bauer, L. de Broglie and M. Courtines. Pp. 29. (Paris: Hermann et Cie., 1933.) 10 fr. (paper covers).

This pamphlet is off-printed from the *Revue d'Électricité et de Mécanique*. E. Bauer contributes an historical introduction and an account of the Compton effect. P. Auger writes on the photoelectric effect, especially at X-ray frequencies, in gases, and M. Courtines describes experiments on the collisions (elastic, inelastic and superelastic) of electrons with gas molecules. The pamphlet concludes with a three-page summary by L. de Broglie on "quanta and wave mechanics."

The treatment, though authoritative, is elementary and non-mathematical, and the discussion is confined exclusively to quantum phenomena in which there is, at least superficially, immediate contact between theory and experiment. As in most French popular scientific writings, the style is very lucid. H. R. R.

* Reviewed in *Proc. Phys. Soc.* **43**, 458 (1931).

Introduction à la Théorie des Groupes et à ses applications à la Physique Quantique, by E. Bauer. Pp. 170. (Paris: Les Presses Universitaires de France.) 40 fr.

For some reason the explicit introduction of the theorems on groups of substitutions into modern quantum theory has called forth a number of books on the subject: the one now under review is more of a treatise on group theory than on quantum theory, and is presumably needed because physicists in general are not familiar with that subject. Let their mathematical teachers, then, take note of the point and see that it is introduced into students' curricula at an earlier stage!

After all, the ideas of group theory are interesting, easy to grasp and useful. The unification they introduce into the subject of differential equations was brought out by the work of Lie long ago, and was put into elementary form in the text book by Page.

As a matter of fact, the whole idea of substitution, and the equivalent idea of transformation, is central to mathematical theory. About half of Forsyth's great treatise on function theory is based on the idea of transformations. When transformation is linear in many variables, we represent it quite naturally by the matrix of the coefficients in the equations of transformation, so that the subject is also closely connected with algebra. Moreover, the multiplication theorem itself for matrices was devised to suit their application to the transformation of variables.

Another non-commutative algebra, the first discovered, is that of quaternions, and it too was based on the idea of transformations, the quaternion being an operator which turns one vector into another. Here we link the notions of geometry to analysis, a co-ordination which can be applied to transformations in general. When the variables are changed from the set $(x_1, x_2, \ldots)$ to the set $(y_1, y_2, \ldots)$ we can imagine them to be cartesian co-ordinates in space. The transformation then figures as a distortion (or remodelling) of the space. For example, if x_1, x_2 are cartesian co-ordinates in a plane, and y_1, y_2 are the polar co-ordinates, then by treating the latter as cartesian co-ordinates the circle $x_1^2 + x_2^2 = a^2$ is turned into the infinite strip $y_1 = \pm a$.

When the idea of geometry is linked to that of linear transformation, objects are moved without distortion (or with only simple strain), but there is a second interpretation of the same transformation. Instead of letting the body move, the axes of reference can do so. This duality in the conceptions appears trivial, but is of an importance which is well brought out in the book under notice.

The English student, if he does not study the original memoirs, has to find his group theory mainly in books devoted to algebra (Chrystal, Bôcher or Turnbull for example) and on differential equations (Ince or Page). A systematic survey of the subject is thus distinctly useful, and the present one, being in French, is easier for the average English reader than are those in German. It is marked by the characteristic lucidity and logical outlook of the French mathematician, and, curiously enough, is printed on good paper. Whilst containing no very recent results, it is more advanced as a treatise on group theory than the English authorities mentioned, and introduces most of the important general results in atomic physics, such as the theory of the Zeeman effect and of exchange degeneracy.

J. H. A.

Actualités Scientifiques et Industrielles, No. 74. *Spectres Moléculaires: Étude des Molécules Diatomiques*, by P. Swings. Pp. 52. (Paris: Hermann et Cie., 1933.) 14 fr.

This recent addition to a well-known series of monographs may be regarded as a continuation of the preceding number (73) by the same author. This accounts for a somewhat abrupt opening, and for the facts that the first table is "Tableau VII" and the first three references are "Weizel, *loc. cit.*...," which is easily identified with *Banden-*

spektren (Leipzig, 1931). Many sections of the monograph, in fact, appear to be adaptations of the corresponding sections of Weizel's book. The subjects briefly discussed include the interactions of molecular rotation and electronic angular momenta, symmetry and antisymmetry in rotational terms, isotope effect, selection rules for term combinations, structures of band-systems and of individual bands of several types, and fluorescence spectra. The selection of material and the treatment are very good, and the booklet should prove extremely useful to those readers, especially those French readers, who do not need the observational data, bibliographies and important details which are to be found in the larger English and German works on this subject. W. J.

Electromagnetic Waves, by F. W. G. White. Pp. viii + 108. Methuen's Monographs on Physical Subjects. (London: Methuen & Co., 1934.) 3*s*.

In this little volume Dr White gives a clear and concise summary of the theory of electric waves, logical but without undue insistence on mathematical rigour, and with a discreet use of the methods of the vector calculus, which should prove of great service to a large number of physics students. The last chapter, on wireless waves in the earth's atmosphere, is an excellent and easy introduction to some of the beautiful work in this field which we owe to Prof. Appleton and his school. H. R. R.

Mass-spectra and Isotopes, by F. W. Aston. Pp. xii + 248. (London: Edward Arnold & Co., 1933.) 15*s*. net.

Dr Aston's first book on *Isotopes* (1922) was mainly concerned with the remarkable and fundamentally important results which had just emerged from its author's beautiful work with the first mass-spectrograph. So definite and so complete within their prescribed limits were these first results that it became immediately evident that the next advance could only be made by still further increasing the accuracy of measurement of isotopic relative masses.

In the volume now under review Aston, who from the first appreciated the need for increased accuracy, gives the whole story of the development of the mass-spectrograph into an instrument of the highest precision. Side by side with the great technical difficulties of introducing the necessary refinements without too much loss of intensity, there have been other problems of a quite different kind but of a similar order of obduracy. These are the problems, very acute for many of the elements, of obtaining suitable beams of mass-rays and getting them into the spectrograph for analysis. The whole investigation is a model of skilled and tenacious experimenting, and it is admirably summarized in this volume; it says much for Dr Aston's power of description that he has been able to depict so many trees without ever obscuring the general view of the wood.

Parts 1 and 2 of the book give a full historical introduction to the whole subject and a detailed account of the work done with the mass-spectrograph and by Bainbridge with his apparatus. Part 3 summarizes our present knowledge of the isotopes of the elements, arranged in their groups. Part 4, "Theoretical and general," which terminates the book, has chapters on the electrical theory of matter, isotope statistics, isotope effects in molecular and atomic spectra (contributed by Dr C. P. Snow), and the separation of isotopes. Full use is made of illustrative diagrams, and the treatment is so clear that the book may quite well be recommended to beginners and to general readers, as well as to the specialists for whom it is indispensable. The appearance of this successor to the *Isotopes* is particularly timely, coming as it does when the events of the past two or three years have so insistently stressed the importance of its subject-matter throughout the whole domain of atomic and nuclear physics. H. R. R.

Faraday's Diary: being the Various Philosophical Notes of Experimental Investigation made by Michael Faraday, D.C.L., F.R.S., during the years 1820–62, volume **3** (May 26, 1836 to Nov. 9, 1839). Pp. xii + 466; volume **4** (Nov. 12, 1839 to June 26, 1847). Pp. xii + 448. (London: G. Bell & Sons.) To be completed in seven volumes. £12. 12*s*. the set.

Apart from the long period during which Faraday was debarred by serious ill health from all sustained experimental research, the record of these two volumes is one of inspired and conspicuously fruitful activity. In volume **3** we find the notes of the classical experiments on "specific inductive capacity," and while Faraday's methods and general results in this field are familiar to the youngest student of electricity, we are enormously enriched by this publication of the detailed notes. There is, of course (to take only one example), a strong interest for all working physicists in such experimental details as those of Faraday's struggles with leaky insulators and soaking dielectrics; what is, however, of the first importance is that we have here the full story of the unfolding of the conception of action through the medium—the "action of continuous particles affecting each other in turn, and not an action at a distance"—as the root cause of electrical phenomena. Nowhere else—not even in the *Experimental Researches*—can we find so perfect a picture of the development of the idea which directed the whole course of electrical theory in the later nineteenth century.

Again, in volume **4** we have the accounts of two further discoveries of fundamental importance—that of the magnetic rotation of light in the "heavy glass" and other media, and that of the universal nature of magnetism. It is worthy of note that these discoveries followed, in rapid succession, hard upon the period of enforced inactivity to which reference has already been made.

The importance of these three major electrical researches is too well known to need stressing, especially in these *Proceedings*. It may, however, not be without point to add that no one who reads the account of them in the *Diary* can fail to appreciate why (as we have it on record in Schuster's *Biographical Fragments*) the perusal of a notice of Faraday's work was sufficient to give to one acute and original mind its first impulsion towards physical science.

Of other notes in these two volumes, the more important series relate to work on the discharge through gases, on voltaic electricity, on the electrification produced by steam jets, and on the liquefaction of gases. In addition there are a number of illuminating speculative passages, disconnected notes on a variety of subjects, mainly electrical, and a detailed account of the experiments performed, with several voluntary assistants and one conscript (the "live gudgeon"), on the long-suffering gymnotus at the Adelaide Gallery.

It remains only to record that the editorship and production are of the high standard which characterized the first two volumes, and that the advent of the new volumes serves to enhance our sense of gratitude to the Managers of the Royal Institution for the service they are performing in giving the *Diary* to the world. H. R. R.

Modern Thermodynamics by the Methods of Willard Gibbs, by E. A. Guggenheim. Pp. xvi + 206. (London: Methuen & Co.) 10*s*. 6*d*. net.

It is probable that no more than a small proportion of the admirers of Gibbs's thermodynamical writings habitually use his methods as working tools for the solution of thermodynamical problems. Here, in a short and well-written treatise, Mr Guggenheim stresses the relations between the chemical potentials and other more recent concepts of physical chemistry, and he builds up logically and elegantly a system of thermodynamical relations which fully displays the utility and power of the methods first developed by Gibbs. His

treatment of the properties of solutions is particularly full, but in contrast with this he dismisses the "third law of thermodynamics" somewhat summarily.

The book should have a definite appeal both to physical chemists and to physicists. This appeal would have been stronger, especially to relatively inexperienced readers, if the author had relaxed the austerity of his design to the extent of including a few numerical examples, representative of the behaviour of actual substances and illustrative of the principles he expounds. If he really anticipates that his book will be read by those who are only "vaguely acquainted with the physical meaning of the first and second laws of thermodynamics," and "not necessarily with their applications" (*Preface*, p. viii), some such concession would clearly not be out of place.

Apart from this real, if minor, defect, the book can be warmly recommended to students of thermodynamics. It is original in design and treatment, and it makes an admirable supplement to the existing texts of a more conventional type. H. R. R.